Solving the constrained output regulation problem using projected dynamical systems

Pietro Lorenzetti[†] (joint work with Prof. George Weiss[‡])

Workshop IMR @ L2S, CentraleSupélec, Université Paris-Saclay - November 16th 2023

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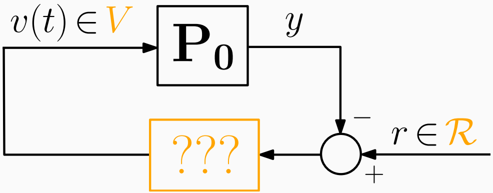
Constrained Output Regulation: Intuition and Motivations

Constrained Output Regulation. Given a plant $\mathbf{P_0}$, a class of references $\mathcal{R}$, and a set V, solve (when possible) the following output regulation problem:



Motivations. Guarantee **safety constraints** and avoid the **windup phenomenon**.

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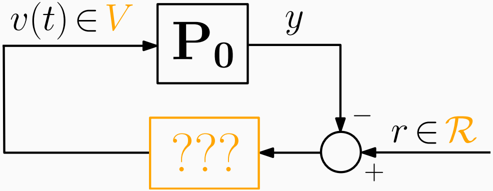
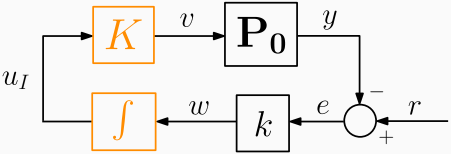


Motivations. Guarantee **safety constraints** and avoid the **windup phenomenon**.

Our focus will be on **integral control**, thus $\mathcal{R}$ = constant references.

Projected Integral Controllers (Lorenzetti & Weiss, 2022)

Classical Integrator



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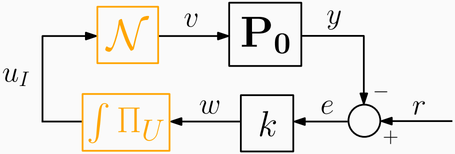
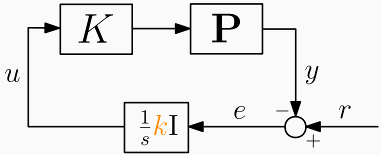


Keywords Safety Constraints, Projected Dynamical Systems, Singular Perturbations.

P. Lorenzetti and G. Weiss. "PI control of stable nonlinear plants using projected dynamical systems," *Automatica*, 2022.

A. Integral Control

A brief review on integral control for linear and nonlinear systems.



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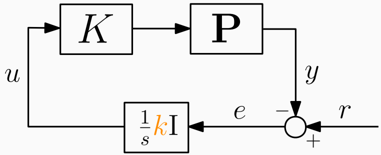
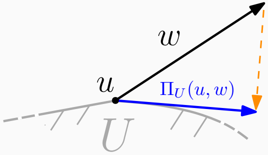


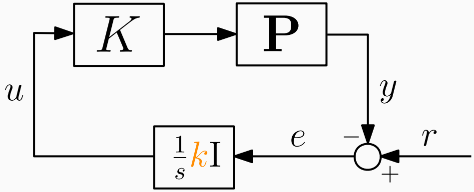
B. Projected Dynamical Systems

How is the operator Π_U defined?



Integral Control

Integral Control for Linear Systems (Davison, 1976; Morari, 1985)



Theorem. Let $\mathbf{P}$ be a linear plant. Assume that:

(a) $\mathbf{P}$ is stable and $\mathbf{P}(0)$ is full row rank.

(b) $\sigma(\mathbf{P}(0)K) \subset \mathbb{C}_+$, i.e., $-\mathbf{P}(0)K$ Hurwitz.

Then, there exists a $\kappa > 0$ such that for all $k \in (0, \kappa)$ the above **closed-loop system is stable** and it exhibits **zero tracking error** for all constant references r.

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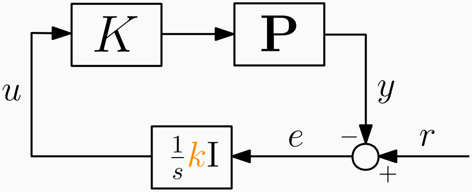

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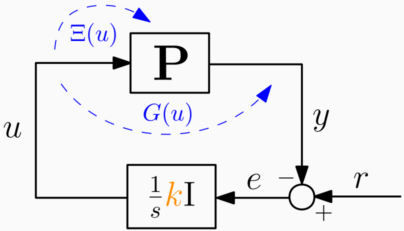
Possible choice. Choose K such that $K = \mathbf{P}(0)^{-1}_{\text{right}}$.

Integral Control for Nonlinear Systems (Desoer & Lin, 1985)



Theorem. Let $\mathbf{P}$ be a nonlinear system. Assume that:

(a) For every constant input $u \in \mathbb{R}^m$, $\Xi(u)$ is a **uniform GES** equilibrium point of $\mathbf{P}$.

(b) The map G is **strictly monotone** for all $v_1, v_2 \in \mathbb{R}^m$.

Then, there exists a $\kappa > 0$ such that if $k \in (0, \kappa)$, then for all **constant references** $r \in \mathbb{R}^m$ the closed-loop system has a **GES** equilibrium point at which $e=0$.

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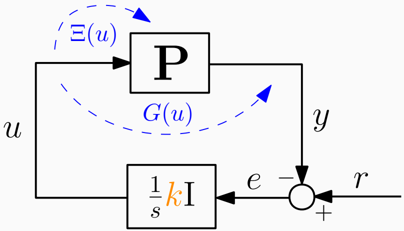

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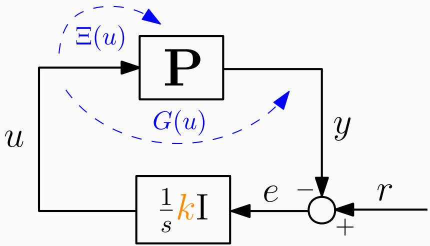
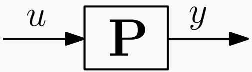
Linear Systems. (a) $\sigma(A) \subset \mathbb{C}_-$; (b) $\mathbf{P}(0) + \mathbf{P}(0)^\top$ **strictly positive definite**.

The Idea Behind: Singular Perturbations



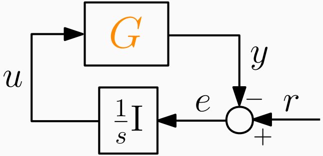
$\Rightarrow$

boundary-layer (fast) system



(a)

reduced (slow) dynamics



(b)

Projected Dynamical Systems

Normal and Tangent Cones

Definition. Let $U \subset \mathbb{R}^m$ be a closed, non-empty, convex set. We define

$$N_U(u) = \{z \in \mathbb{R}^m \mid \langle z, v - u\rangle \leq 0 \quad \forall\, v \in U\} \qquad \forall\, u \in U,$$
$$N_U(u) = \emptyset \qquad \forall\, u \notin U.$$

$$T_U(u) = \{s \in \mathbb{R}^m \mid \langle z, s\rangle \leq 0 \quad \forall\, z \in N_U(u)\} \qquad \forall\, u \in U,$$
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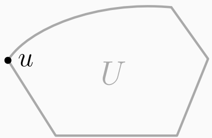
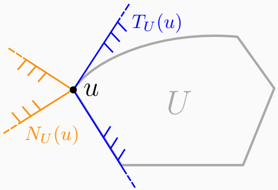
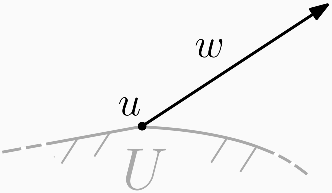
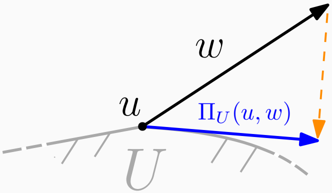




Definition. Let $U \subset \mathbb{R}^m$ be a closed, non-empty, convex set. For every $u \in U$ and $w \in \mathbb{R}^m$, we define

$$\Pi_U(u, w) = \arg\min_{v \in T_U(u)} \|w - v\|.$$





Projected Dynamical Systems - Definition of Solutions

Definition. Let $F \in C(\mathbb{R}^m; \mathbb{R}^m)$ and $U \subset \mathbb{R}^m$ be non-empty, closed and convex. A function $u \in W^{1,1}((0,\tau); \mathbb{R}^m)$ that satisfies

$$\dot{u}(t) = \Pi_U(u(t), -F(u(t))) \qquad u(0) = u_0 \tag{PDS}$$

for a.e. $t \in (0,\tau)$ is called a **Carathéodory solution** to (PDS).

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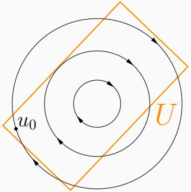

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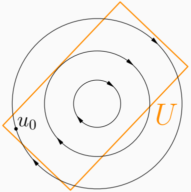

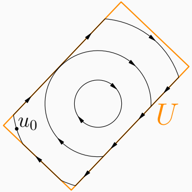

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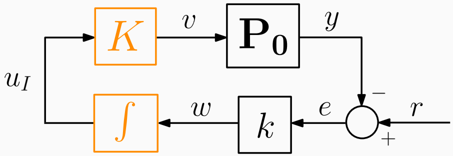

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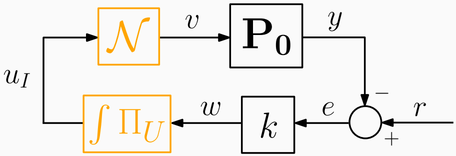

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What is the control problem
that we address?

2. Stability Analysis

How to derive stability and tracking
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in a grid-connected synchronverter?

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Control Problem Formulation

The **nonlinear plant $\mathbf{P_0}$** to be controlled is described by:

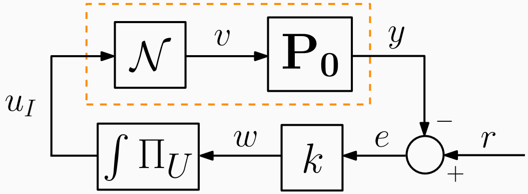
$$\dot{x} = f_0(x, v), \qquad y = g(x),$$

with $f_0 \in C^2(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$, $g \in C^1(\mathbb{R}^n; \mathbb{R}^p)$, with $m \geq p$.

Control Objective

The control objective is to make the output signal y tracks a **constant reference** signal $r \in Y \subset \mathbb{R}^p$, while making sure that the plant input signal v remains in a **desired compact set** $V \subset \mathbb{R}^m$ (e.g., determined by operational constraints).

Closed-Loop System - Equations



The **closed-loop system** is described by

$$\dot{x} = f_0(x, \mathcal{N}(u_I)), \quad \dot{u}_I = \Pi_U(u_I, k(r - g(x))), \tag{CL}$$

where $U \subset \mathbb{R}^p$ is compact and convex, $\mathcal{N} \in C^2(\mathbb{R}^p, \mathbb{R}^m)$, $V = \mathcal{N}(U)$, and $k > 0$.

Closed-Loop Stability Analysis

Assumption 1

Assumption 1. There exists an open domain $\mathcal{V} \subset \mathbb{R}^m$ and $\Xi \in C^1(\mathcal{V}; \mathbb{R}^n)$ such that

$$f_0(\Xi(v), v) = 0 \qquad \forall\, v \in \mathcal{V},$$

and the equilibrium points $\{\Xi(v) \,\big|\, v \in \mathcal{V}\}$ are **uniformly locally exponentially stable**.

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This means that there exist $\varepsilon_0 > 0$, $\lambda > 0$ and $\rho \geq 1$ such that for each **constant input** $v_0 \in \mathcal{V}$, the following holds: If $\|x(0) - \Xi(v_0)\| \leq \varepsilon_0$, then for every $t \geq 0$,

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Linear Systems. Let $\mathbf{P}_0$ be linear ($\dot{x} = Ax + Bv$, $y = Cx$), Assumption 1 reduces to A being **stable**. The function Ξ is given by

$$\Xi(v) = (-A)^{-1} B v.$$

Assumption 2

Assumption 2. $\mathbf{P_0}$ satisfies Assumption 1. Let $G(v) := g(\Xi(v)) \in C^1(\mathcal{V}; \mathbb{R}^p)$.There exist an open set $\mathcal{U} \subset \mathbb{R}^p$, a function $\mathcal{N} \in C^2(\mathcal{U}, \mathcal{V})$, and $\mu > 0$ such that

$$\langle G(\mathcal{N}(u_1)) - G(\mathcal{N}(u_2)), u_1 - u_2 \rangle \geq \mu \|u_1 - u_2\|^2$$

for all $u_1, u_2 \in \mathcal{U}$, i.e., $G \circ \mathcal{N}$ is **strictly monotone**.

Choice of $\mathcal{N}$. When possible, we suggest the choice $\mathcal{N} = G^{-1}_{\text{right}}$, so that $\mathcal{U} = G(\mathcal{V})$.

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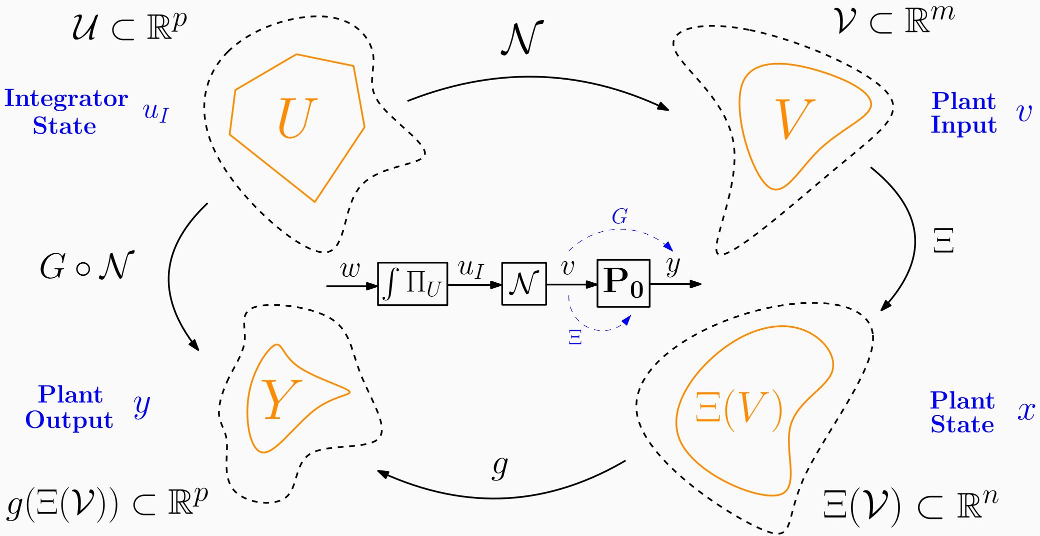
Linear Systems. Let $\mathbf{P}_0$ be linear and $\mathcal{N} \in \mathbb{R}^{m \times p}$, Assumption 2 reduces to $\mathbf{P}_0(0)\mathcal{N} + (\mathbf{P}_0(0)\mathcal{N})^\top$ being **strictly positive definite**. The function G is given by

$$G(v) = C\Xi(v) = C(-A)^{-1}Bv = \mathbf{P}_0(0)v.$$

Mappings Recap



Main Stability Theorem

Notation. We denote $Y = G(\mathcal{N}(U))$, and, for any $r \in Y$, we define

$$u_r := (G \circ \mathcal{N})^{-1}(r) \qquad x_r := \Xi(\mathcal{N}(u_r)).$$

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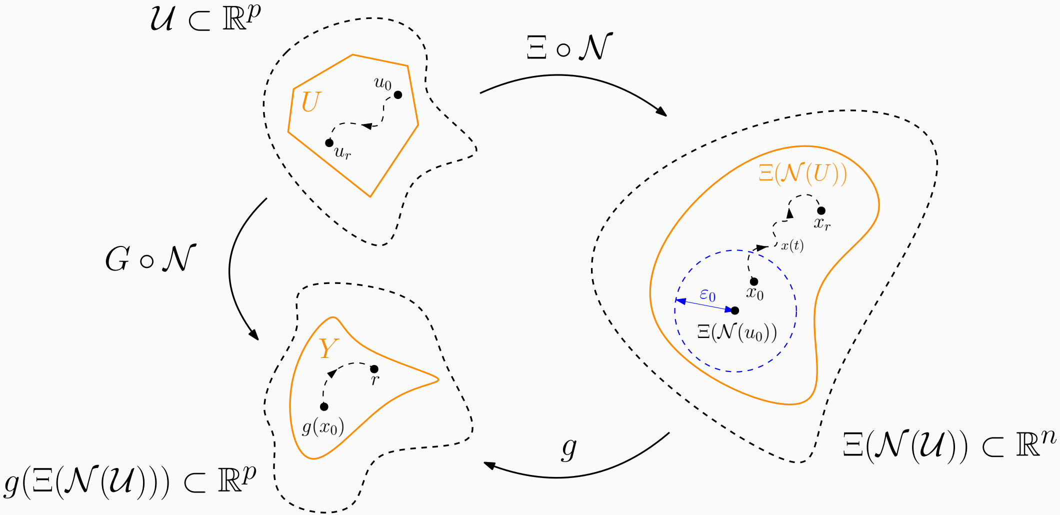
$$u_r := (G \circ \mathcal{N})^{-1}(r) \qquad x_r := \Xi(\mathcal{N}(u_r)).$$

Theorem 1. Consider the closed-loop system (CL), where $\mathbf{P_0}$ satisfies Assumption 2. Then there exists a $\kappa > 0$ such that if the gain $k \in (0, \kappa]$, then for any $r \in Y$, $(\Xi(\mathcal{N}(u_r)), u_r)$ is a (locally) **exponentially stable equilibrium point** of the closed-loop system (CL), with state space $X = \mathbb{R}^n \times \mathcal{U}$. If the initial state $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in X$, of the closed-loop system satisfies $u_0 \in U$ and $\|x_0 - \Xi(\mathcal{N}(u_0))\| \leq \varepsilon_0$, then

$$x(t) \to \Xi(\mathcal{N}(u_r)), \qquad u_I(t) \to u_r, \qquad y(t) \to r,$$

and this convergence is at an **exponential rate**.

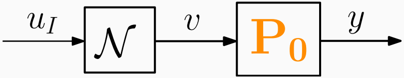
Intuition of the Result



Intuition of the Proof - Singular Perturbations

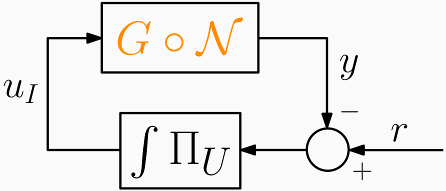
Reduced (slow) Model

Boundary-Layer (fast) System



$$\dot{x} = f_0(x, \mathcal{N}(u_I)),$$

where $u_I \in U$ is **fixed**.

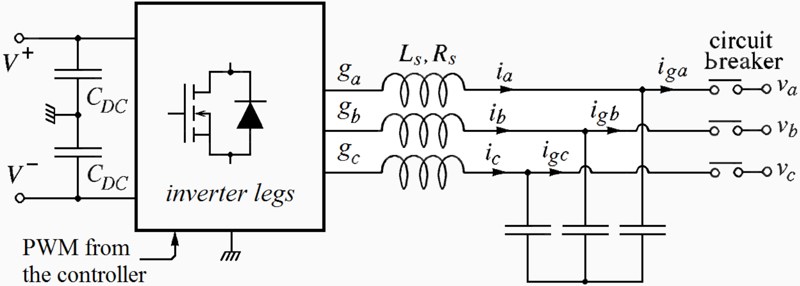


$$\frac{\mathrm{d}u_I}{\mathrm{d}s} = \Pi_U(u_I, r - G(\mathcal{N}(u_I))),$$

where $s = k \cdot t$ **slow** time-scale.

Constrained Power Regulation for a Grid-Connected Synchronverter

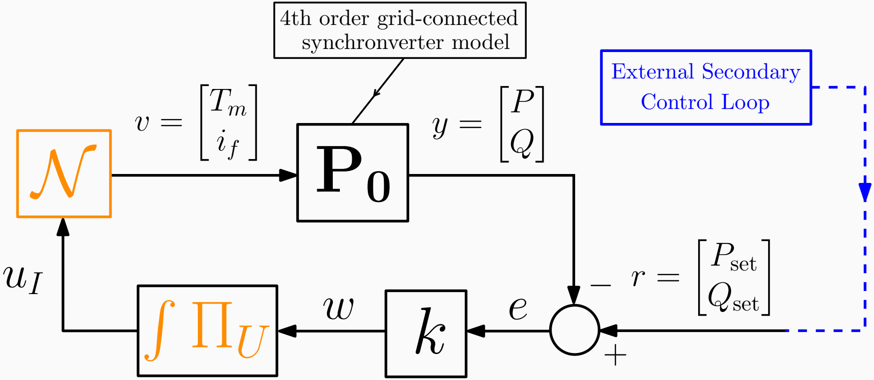
A Grid-Connected Inverter with an LC Filter



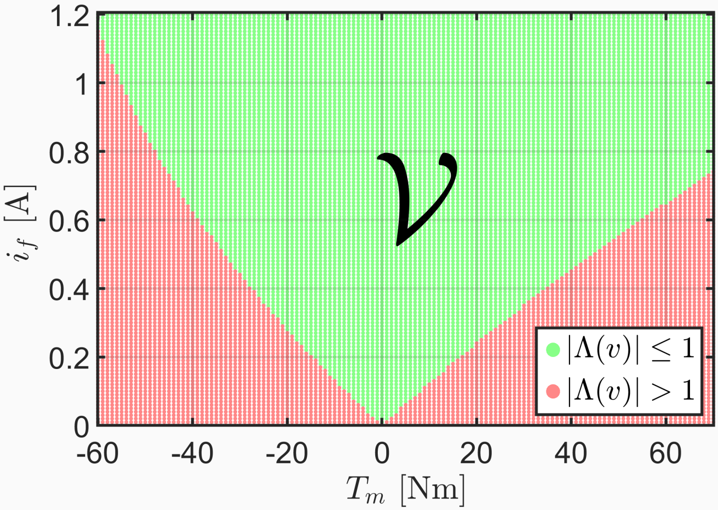
The above inverter is controlled as a **synchronverter**, i.e., as a synchronous generator.

P. Lorenzetti, Z. Kustanovich, S. Shivratri and G. Weiss, "The equilibrium points and stability of grid-connected synchronverters," *IEEE Transactions on Power Systems*, 2022.

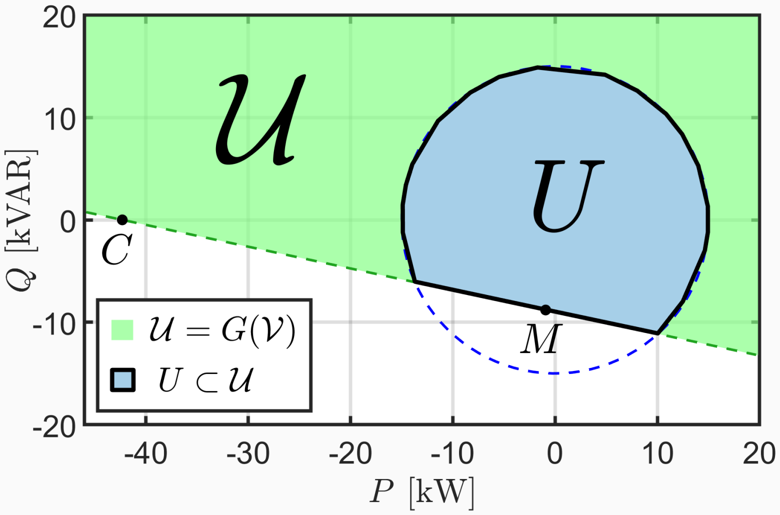
Closed-Loop Control Scheme



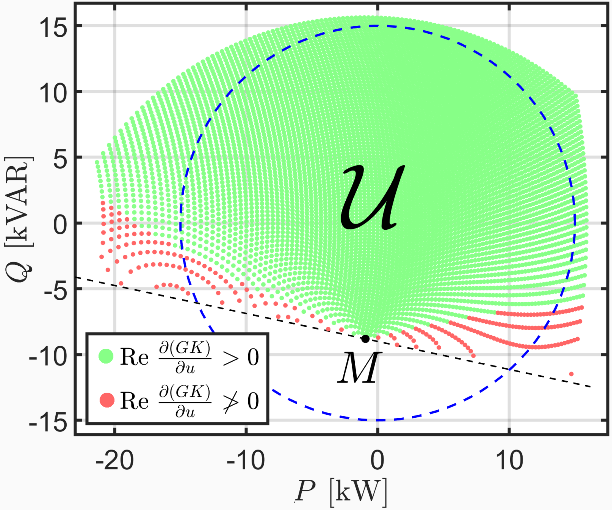
Assumption 1 - The set $\mathcal{V}$ ($\Xi : \mathcal{V} \to \mathbb{R}^4$)



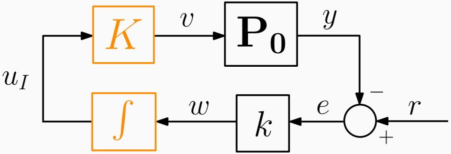
Assumption 2 - The sets $\mathcal{U}$ and U when $\mathcal{N} = G_{\text{right}}^{-1}$



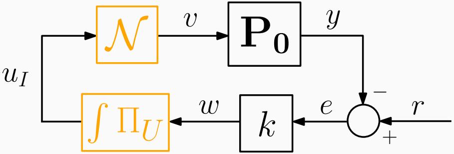
Assumption 2 - The sets $\mathcal{U}$ when $\mathcal{N} = K \in \mathbb{R}^{2\times 2}$



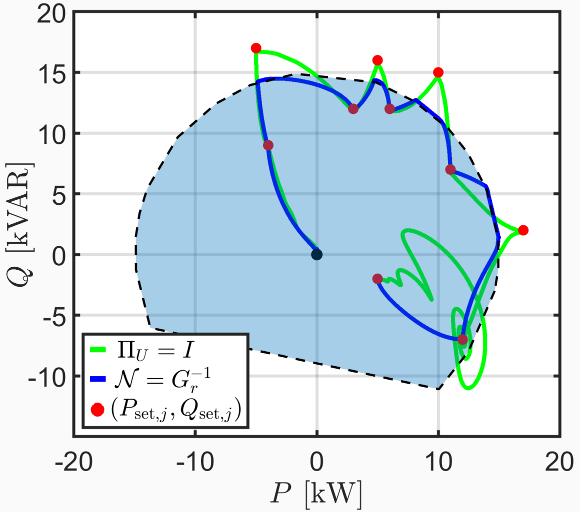
Classical I control loop



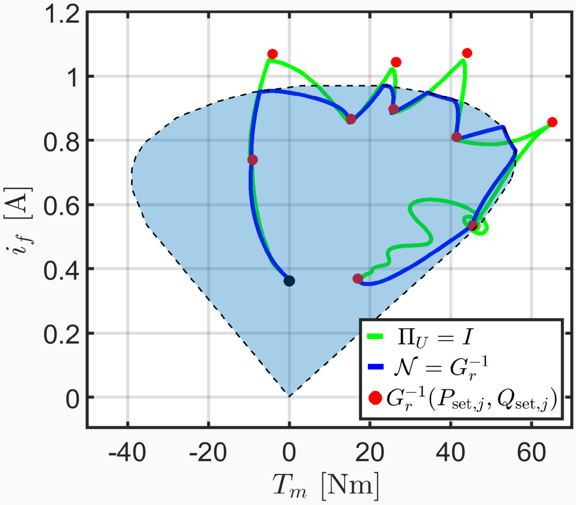
Projected I control loop



Numerical Results - The Integrator State u_I and the Set U



Numerical Results - The Plant Input v and the Set V



Conclusion & Perspectives

Conclusion

Projected Integrator
Projected Systems Π_U
Safety Constraints

Stability Analysis
Singular Perturbations
Gain $\mathcal{N} = G_{\text{right}}^{-1}$

Power Systems
Power Regulation for a
Synchronverter

Possible Future Directions

Time-Varying Constraints. In some power systems applications, we would like

$$\dot{u}_I = \Pi_{U(t)}(u_I, w).$$

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Weaker Notions of Stability. What can we say on aGAS systems?

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Weaker Notions of Stability. What can we say on aGAS systems?

Infinite-Dimensional Systems. Preliminary results for well-posed SISO linear systems.

Thanks for your attention!