Harmonic-based modeling and control

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Context and motivations

- Fact: Electrical actuation chains are subject to undesirable harmonic disturbances and/or non-linear resonance phenomena.

  Out of control because of harmonics - an analysis of the harmonic response of an inverter locomotive, E. Mollersted et al. IEEE Control Systems Magazine, 2000.

  Swiss locomotive stopped due to high harmonic currents:

  - Instabilities caused by interactions between systems
  - The modeling of the electrical chain was insufficient.

Harmonic control allows to:

- Design stabilizing control for periodic systems
- Take into account and cancel undesirable harmonic content

State of the art of harmonic modeling

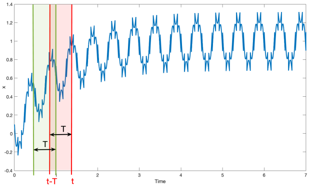
Earlier and seminal papers (mainly developed in Electrical Engineering):

- General framework:
  - Generalized State-Space Averaging (GSSA) [S. R. Sanders & al. (1991)]
  - Dynamic Phasors (DP) [P. Mattavelli, G. C. Verghese & A. M. Stankovic (1997)]
- Linear Time Periodic (LTP) systems
  - Sur les équations linéaires à coefficients périodiques [G. Floquet, (1883)].
  - Extended Harmonic Domain (EHD) [M. Madrigal (2001)]
  - Dynamic Harmonic Domain (DHD) [J. J. Chavez & A. Ramirez (2008)]
  - Harmonic State Space (HSS) [N. M. Wereley (1990)]

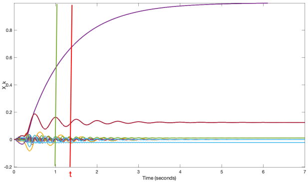
In [Blin & al. (EJC 2020)], we show that all these harmonic modeling methods are equivalent and are based essentially on the same tool:

`a Sliding Fourier Decomposition (SFD) over a window of length` T.

Sliding Fourier Decomposition over a window of length T



Signal x



Phasors $X = \mathcal{F}(x)$

More precisely:

$$\begin{aligned} \mathcal{F} : L^2_{loc}(\mathbb{R}, \mathbb{C}) &\to L^\infty_{loc}(\mathbb{R}, \ell^2(\mathbb{C})) \\ x &\mapsto X \end{aligned}$$

where the components of the time varying infinite sequence $X = (\cdots, X_{-1}, X_0, X_1, \cdots)$ are defined by:

$$X_{\mathbf{k}}(t) := \frac{1}{T} \int_{t-T}^{t} x(\tau) e^{-j\omega \mathbf{k} \tau} d\tau \text{ (time varying Fourier coef.) with } \omega := \frac{2\pi}{T}.$$

$X_{\mathbf{k}}$ is called $k-th$ phasor (harmonic) of X.

If $x = (x_1, x_2, \cdots, x_n)$ is a vector function then $X := (\mathcal{F}(x_1), \mathcal{F}(x_2), \cdots, \mathcal{F}(x_n))$

How to determine a Harmonic Model ?

Consider a differential equation

$$\dot{x} = f(t, x) \tag{1}$$

Formally, its harmonic model is determined by:

$$\dot{X} = \mathcal{F}(f(t, x)) - \mathcal{N}X \tag{2}$$

where $\mathcal{N}$ is a diagonal operator.

But there are important issues not addressed in the previous literature:

1. How to reconstruct exactly x, if it exists, from X ? (inverse and functional space invoked)
2. Under which conditions we have: (1) $\Leftrightarrow$ (2) ? (bijection)

These questions are essential for analysis and synthesis purposes in the harmonic domain.

For example, System (2) has trajectories that have no counterpart in (1).
Thus, if we design a harmonic control U, there is no guarantee that $u = \mathcal{F}^{-1}(U)$ exists!

Outline

1. Harmonic modeling - a mathematical framework
2. Analysis and Control design
   1. Solving harmonic Lyapunov, Sylvester and Riccati equations
   2. Solving harmonic Toeplitz Block LMIs (TBLMIs)
3. Applications
   1. Rejection of harmonic disturbances on a three-phase rectifier bridge (SAFRAN)
   2. Optimal state feedback design for LTP systems by solving TBLMIs

Mathematical framework

Bijection between functional spaces

Theorem 1 (Coincidence Condition)

There exists a representative $x \in L^2_{loc}(\mathbb{R}, \mathbb{C}^n)$ of X, with $X \in L^\infty_{loc}(\mathbb{R}, \ell^2(\mathbb{C}^n))$, if and only if, X is absolutely continuous and fulfills for any $k \in \mathbb{Z}$:

$$\dot{X}_k(t) = \dot{X}_0(t) e^{-\mathrm{j}\omega k t} \text{ a.e.} \tag{3}$$

X is said to belong to $H \subset C^{ac}(\mathbb{R}, \ell^2(\mathbb{C}^n))$

Reconstruction formula

Theorem 2 (punctual convergence)

If $x \in C^1_{pw}$(or C^0_{pw} with bounded variations), then the reconstruction formula is provided by:

$$x(t) = \sum_{p=-\infty}^{+\infty} X_p(t) e^{\mathrm{j}\omega p t} + \frac{T}{2} \dot{X}_0(t), \tag{4}$$

except at points of discontinuity of x for which left and right limits exist. In addition, if $x \in C^0$, the equality (4) holds everywhere.

Proofs: "Necessary and sufficient conditions for harmonic control" [IEEE TAC 2022].

Harmonic systems

Consider nonlinear dynamical systems described by:

$$\dot{x} = f(t, x), \quad x(0) := x_0 \tag{5}$$

Theorem 3

Under weak assumptions, x is a solution of the differential equation (5) in the Carathéodory sense, if and only if, $X = \mathcal{F}(x) \in H$ is a solution of:

$$\dot{X} = \mathcal{F}(f(t, x)) - \mathcal{N}X, \quad X(0) := \mathcal{F}(x)(0) \tag{6}$$

with $\mathcal{N} := Id_n \otimes diag(\mathrm{j}\omega k,\ k \in \mathbb{Z})$. (Infinite dimensional system !)

Proofs: "Necessary and sufficient conditions for harmonic control" [IEEE TAC 2022].

Interest for analysis and control:

- $T-$periodic systems becomes time invariant in harmonic domain
  → All time invariant control design methods can be a priori applied
- A $T-$periodic trajectory corresponds to an equilibrium in the harmonic domain

How to determine $\mathcal{F}(f(t,x))$?

- Define Toeplitz Block (TB) transformation $\mathcal{T}(\cdot)$ of matrix function $A_{n\times m} := (a_{ij})$

by: $$\mathcal{A} := \mathcal{T}(A) = \begin{bmatrix} \mathcal{A}_{11} & \dots & \mathcal{A}_{1m} \\ \vdots & & \vdots \\ \mathcal{A}_{n1} & \dots & \mathcal{A}_{nm} \end{bmatrix}$$

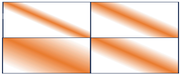


Figure: 2×2*TB representation*

with $$\mathcal{A}_{ij} = \mathcal{T}(a_{ij}) := \begin{bmatrix} \ddots & & \vdots & & \iddots \\ & a_{ij,0} & a_{ij,-1} & a_{ij,-2} & \\ \cdots & a_{ij,1} & a_{ij,0} & a_{ij,-1} & \cdots \\ & a_{ij,2} & a_{ij,1} & a_{ij,0} & \\ \iddots & & \vdots & & \ddots \end{bmatrix}, (\infty \times \infty \text{ Toeplitz matrix})$$

where $a_{ij,k}$, $k \in \mathbb{Z}$ refers to the phasors (Fourier coef.) of a_{ij}.

Property 1

- *matrix-vector product:* $\mathcal{F}(Ax) = \mathcal{T}(A)\mathcal{F}(x) = \mathcal{A}X$
- *matrix-matrix product:* $\mathcal{T}(AB) = \mathcal{T}(A)\mathcal{T}(B) = \mathcal{A}\mathcal{B}$

Useful when $f(t,x)$ defines a time-periodic polynomial systems !

Property 2

- $A \in L^\infty([0\ T])$ *if and only if* $\mathcal{A} := \mathcal{T}(A)$ *is a constant and bounded on* ℓ^2 *operator*

$$\|\mathcal{A}\|_{\ell^2} := \sup_{\|X\|_{\ell^2}=1} \|\mathcal{A}X\|_{\ell^2} = \|A\|_{L^\infty}$$

Example : Harmonic model of LTP systems

Consider $A \in L^2([0\ T])$ and $B \in L^\infty([0\ T])$ T-periodic matrices.
x is the unique solution associated with $u \in L^2_{loc}$ of the LTP system,

$$\dot{x} = A(t)x(t) + B(t)u(t), \quad x(0) := x_0 \tag{7}$$

if and only if $X = \mathcal{F}(x) \in H$ is the unique solution associated with $U := \mathcal{F}(u) \in H$ of the LTI system

$$\dot{X} = (\mathcal{A} - \mathcal{N})X + \mathcal{B}U, \quad X(0) := \mathcal{F}(x)(0) \tag{8}$$

where $\mathcal{A} := \mathcal{T}(A)$ and $\mathcal{B} := \mathcal{T}(B)$.

1. How to perform a stability analysis?
2. How to design a state feedback $U := -\mathcal{K}X$?

Harmonic Lyapunov, Sylvester and Riccati Equations

Theorem 4 (Harmonic Lyapunov equation)

Assume that $A \in L^2([0\ T], \mathbb{R}^{n\times n})$ is a T-periodic matrix function and let $Q \in L^\infty([0\ T])$ be a T-periodic symmetric and positive definite matrix function. P is the unique T-periodic symmetric positive definite solution of the periodic Lyapunov differential equation:

$$\dot{P}(t) + A'(t)P(t) + P(t)A(t) + Q(t) = 0, \tag{9}$$

if and only if $\mathcal{P} = \mathcal{T}(P)$ is the unique hermitian and positive definite solution of the algebraic Lyapunov equation:

$$(\mathcal{A} - \mathcal{N})^* \mathcal{P} + \mathcal{P}(\mathcal{A} - \mathcal{N}) + \mathcal{Q} = 0, \tag{10}$$

where $\mathcal{Q} := \mathcal{T}(Q)$ is hermitian positive definite and $\mathcal{A} := \mathcal{T}(A)$. Moreover, $\mathcal{P}$ is a bounded operator on ℓ^2.

In practice, solving an infinite dimensional problem implies to solve a truncated finite-dimensional one with a consistent scheme.

Solving Harmonic Lyapunov equation

Main idea : Take advantage of the Toeplitz structure of the infinite dimensional problem and solve a finite-dimensional r-truncated version

Theorem 5 (Infinite dimensional solution)

Assume that $A \in L^\infty([0\ T], \mathbb{R}^{n\times n})$ and $(\mathcal{A}-\mathcal{N})$ is invertible. The phasor $\mathbf{P} := \mathcal{F}(P)$ associated with the solution $\mathcal{P} := \mathcal{T}(P)$ of the infinite-dimensional harmonic Lyapunov equation is given by:

$$col(\mathbf{P}) = -(Id_n \otimes (\mathcal{A}-\mathcal{N})^* + Id_n \circ \mathcal{A}^*)^{-1} col(\mathbf{Q}) \tag{11}$$

where $Id_n \circ \mathcal{A} := \begin{pmatrix} Id_n \otimes \mathcal{A}_{11} & \cdots & Id_n \otimes \mathcal{A}_{1n} \\ \vdots & \ddots & \vdots \\ Id_n \otimes \mathcal{A}_{n1} & \cdots & Id_n \otimes \mathcal{A}_{nn} \end{pmatrix}$ *and* $\mathbf{Q} := \mathcal{F}(Q)$.

Definition 6 (r-truncation operator Π_r)

- for a phasor vector $X := \mathcal{F}(x)$: $\Pi_r(X) := (X_{1,-r:r}, X_{2,-r:r}, \cdots, X_{n,-r:r})$
- for $n \times m$ infinite-dimensional TB matrix $\mathcal{A} := \mathcal{T}(A)$

$$\Pi_r(\mathcal{A}) := \begin{bmatrix} \Pi_r(\mathcal{A}_{11}) & \dots & \Pi_r(\mathcal{A}_{1m}) \\ \vdots & & \vdots \\ \Pi_r(\mathcal{A}_{n1}) & \dots & \Pi_r(\mathcal{A}_{nm}) \end{bmatrix} \qquad n(2r+1)\times m(2r+1)$$

where $\Pi_r(\mathcal{A}_{ij}) := \begin{pmatrix} a_{ij,0} & \cdots & a_{ij,-2r} \\ \vdots & \ddots & \vdots \\ a_{ij,2r} & \cdots & a_{ij,0} \end{pmatrix}$ $(2r+1)\times(2r+1)$ principal submatrix of $\mathcal{A}_{ij}$.

Solving Harmonic Lyapunov equation

- Define for any given r, the r−truncated solution $\tilde{\mathbf{P}}_r$ to harmonic Lyapunov equation as

$$\mathrm{col}(\tilde{\mathbf{P}}_r) := -(Id_n \otimes \Pi_r(\mathcal{A}-\mathcal{N})^* + Id_n \circ \Pi_r(\mathcal{A})^*)^{-1}\mathrm{col}(\Pi_r(\mathbf{Q})) \qquad (12)$$

This is a linear problem of dimension $n^2(2r+1)$!

Theorem 7 (Consistency)

For any given $\epsilon > 0$, there exists r_0 such that for any $r \geq r_0$:

$$\|P - \tilde{P}_r\|_{L^\infty} = \|\mathcal{P} - \tilde{\mathcal{P}}_r\|_{\ell^2} < \epsilon$$

with $\mathcal{P} := \mathcal{T}(P)$ and $\tilde{\mathcal{P}}_r := \mathcal{T}(\tilde{P}_r)$.

- Similar results for Harmonic Sylvester and Riccati equations as well as a Spectral Characterization (Floquet Factorization revisited) can be found in "Solving Infinite-Dimensional Harmonic Lyapunov and Riccati equations", [IEEE TAC 2023] and in "Harmonic pole placement" [CDC 2022].

What about harmonic LMIs ?

- How to solve the infinite dimensional harmonic Lyapunov inequality:

$$(\mathcal{A}-\mathcal{N})^*\mathcal{P}+\mathcal{P}(\mathcal{A}-\mathcal{N})+\mathcal{Q}<0 \tag{13}$$

and more generally a semidefinite optimization problem:

$$\min_{\mathcal{P}=\mathcal{P}^*>0} Tr_0(\mathcal{P}) \tag{14}$$

$$\mathcal{L}(\mathcal{P};\mathcal{A}_s, s\in\mathbb{S})<0,$$

where

1. $Tr_0(\mathcal{P})=\sum_{i=1}^n P_{ii,0}$ (average value of $tr(P(t))$ over a period T)
2. $\mathbb{S}$ is a finite set of subscripts,
3. $\mathcal{A}_s := \mathcal{T}(A_s)$ refers to the entries with $A_s\in L^\infty([0\ T])\equiv\mathcal{A}_s$ bounded on ℓ^2.
4. $\mathcal{P}$ and $\mathcal{L}(\mathcal{P};\mathcal{A}_s, s\in\mathbb{S})$ are bounded on ℓ^2

Remark: $tr_0(\mathcal{M}^*\mathcal{M})^{\frac{1}{2}}$ is a norm that satisfies: $\|\mathcal{M}\|_{\ell^2}\leq tr_0(\mathcal{M}^*\mathcal{M})^{\frac{1}{2}}\leq\sqrt{n}\|\mathcal{M}\|_{\ell^2}$

Main ideas

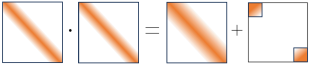
- To approximate $\mathcal{P}$, can we solve for a given r:

$$\Pi_r \left[(\mathcal{A} - \mathcal{N})^* \mathcal{P} + \mathcal{P}(\mathcal{A} - \mathcal{N}) + \mathcal{Q} \right] < 0 \ ? \qquad (15)$$

Two main difficulties:

1. Difficulty 1: How to determine $\Pi_r(\mathcal{AB})$?

   Product of finite dimensional Toeplitz matrices is not Toeplitz

   $$\Pi_r(\mathcal{A})\Pi_r(\mathcal{B}) = \Pi_r(\mathcal{AB}) + E(\mathcal{A}, \mathcal{B})$$

   1. Explicit expression of $E(\mathcal{A}, \mathcal{B})$ is known (implies Hankel Bloc matrices defined from A, B).
   2. $E(\mathcal{A}, \mathcal{B})$ can be exactly computed only if $\mathcal{A}$ or $\mathcal{B}$ are banded TB.

   Solution: Band and Truncate (possible since $A_k \to 0, |k| \to +\infty$)

2. Difficulty 2: If all TB matrices in (15) are now replaced by banded TB matrices, how does the solution obtained compare with the original solutions?

   - Non-uniqueness of the solution complicates the problem

   Solution: Convex optimization problems have generally an unique solution

Solving TBLMI

- Define $\mathcal{A}_{b(p)}$ the $p-$banded version of $\mathcal{A}$ obtained by deleting all its phasors of order higher than p
- Define the finite dimensional problem:
  For given p, q, r, solve

$$\min_{\mathcal{P}=\mathcal{P}^*} Tr_0(\mathcal{P}), \quad \Pi_r(\mathcal{P}) > 0 \tag{16}$$
$$\Pi_r[\mathcal{L}(\mathcal{P}; \mathcal{A}_{s_{b(p)}}, s \in \mathbb{S})] < 0, \quad \mathcal{P}_{ij,k} = 0, \ |k| > q$$

  This is a $\frac{n(n+1)}{2}(2q+1)$ dimensional problem if $\mathcal{P}$ is $n \times n$ TB.

Theorem 8 (Consistency)

Assume Problem (14) has a unique solution $\hat{\mathcal{P}}$ bounded on ℓ^2. For any $\epsilon > 0$, there exist p, q and r_0 such that for any $r > r_0$, the solution $\hat{\mathcal{P}}_{p,q,r}$ to (16) satisfies:

$$\|\hat{P}_{p,q,r} - \hat{P}\|_{L^\infty} = \|\hat{\mathcal{P}}_{p,q,r} - \hat{\mathcal{P}}\|_{\ell^2} < \epsilon. \tag{17}$$

with $\hat{\mathcal{P}} := \mathcal{T}(\hat{P})$ and $\hat{\mathcal{P}}_{p,q,r} := \mathcal{T}(\hat{P}_{p,q,r})$.

Proofs: "On solving infinite-dimensional Toeplitz Block LMIs" [CDC 2023].

Case study: Harmonic control of three-phase rectifier bridge (Safran)

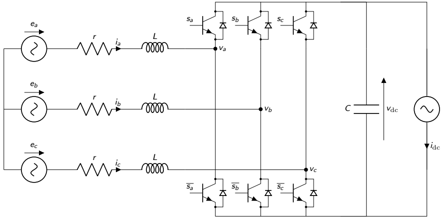


Figure: Grid tied AC/DC converter with load represented as current source

Bilinear system equations (in balanced mode and abc frame representation):

$$\dot{x} = Ax + G(x)d + Bv \qquad (18)$$

- State: $x = (i_{abc}, v_{dc})$,
- Control: $d = d_{abc}$ (duty cycle)
- Input: $v = (e_{abc}, i_{dc})$,

with

$$A = \begin{bmatrix} -\frac{r}{L} I_3 & 0 \\ 0 & 0 \end{bmatrix}, \qquad G(x) = \begin{bmatrix} \frac{C_{33}}{L} v_{dc} \\ \frac{i_{abc}{}'}{C} \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{I_3}{L} & 0 \\ 0 & -\frac{1}{C} \end{bmatrix} \qquad C_{33} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Assume state x is measured.

Control objectives

1. Primary objectives: under T-periodic load or grid perturbation:
   - Maintain the DC bus voltage mean value at a given reference $v_{dc_{ref}}$,
   - Maximize power factor: i_q mean value must be maintained to 0.

2. Secondary objectives: Reduce Total Harmonic Distortion (THD) on i_{abc} to avoid AC grid pollution

   Retained scenario: Reject (2, 4, 5, 7)-th harmonics on i_{abc} due to 3-rd and 6-th harmonic perturbations on i_{dc} (load perturbations)

Park's transformation: $i_{dq0} = \begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos(\omega t) & \cos(\omega t - \frac{2\pi}{3}) & \cos(\omega t + \frac{2\pi}{3}) \\ -\sin(\omega t) & -\sin(\omega t - \frac{2\pi}{3}) & -\sin(\omega t + \frac{2\pi}{3}) \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} i_{\mathrm{abc}}$

where ω is the pulsation of the source grid voltage

Harmonic model of the AC/DC Converter

- Associated harmonic model is an infinite dimensional bilinear system given by:

$$\dot{X} = (\mathcal{A} - \mathcal{N})X + \mathcal{G}(X)D + \mathcal{B}V \qquad (19)$$

with

$$X = (I_a, I_b, I_c, V_{\mathrm{dc}}) = \mathcal{F}(x)$$
$$D = (D_a, D_b, D_c) = \mathcal{F}(d_{\mathrm{abc}})$$
$$V = (E_a, E_b, E_c, I_{\mathrm{dc}}) = \mathcal{F}(e_{abc}, i_{\mathrm{dc}})$$

and where

$$\mathcal{A} := \begin{bmatrix} -\frac{r}{L} I_3 \otimes \mathcal{I} & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} \frac{I_3 \otimes \mathcal{I}}{L} & 0 \\ 0 & -\frac{\mathcal{I}}{C} \end{bmatrix}, \quad \mathcal{G}(X) := \begin{bmatrix} \frac{C_{33} \otimes \mathcal{V}_{\mathrm{dc}}}{L} \\ \frac{\mathcal{I}_{\mathrm{abc}}{}^{*}}{C} \end{bmatrix}$$

with $\mathcal{V}_{\mathrm{dc}} = \mathcal{T}(v_{\mathrm{dc}})$, $\mathcal{I}_{\mathrm{abc}} = \mathcal{T}(i_{\mathrm{abc}})$.

Harmonic stabilizing control

- Consider an equilibrium (X^e, D^e, V^e) given by:

$$0 = (\mathcal{A} - \mathcal{N})X^e + \mathcal{G}(X^e)D^e + \mathcal{B}V^e \tag{20}$$

- The dynamic of the error $\overline{X} = X - X^e$ satisfies:

$$\dot{\overline{X}} = (\mathcal{A} + \mathcal{A}(\mathcal{D}^e) - \mathcal{N})\overline{X} + \mathcal{G}(X)\overline{D} \tag{21}$$

where $\overline{D} = D - D^e$ and $\mathcal{A}(\mathcal{D}^e) = \begin{bmatrix} 0 & \frac{(C_{33} \otimes \mathcal{I})\mathcal{D}^e}{L} \\ \frac{\mathcal{D}^{e*}}{C} & 0 \end{bmatrix}$ with $\mathcal{D}^e = \mathcal{T}(D^e)$.

Theorem 9 (Stabilizing control)

Assume $(\mathcal{A} + \mathcal{A}(\mathcal{D}^e) - \mathcal{N})$ is Hurwitz. Consider $\mathcal{P}$ solution to the Lyapunov equation:

$$(\mathcal{A} + \mathcal{A}(\mathcal{D}^e) - \mathcal{N})^*\mathcal{P} + \mathcal{P}(\mathcal{A} + \mathcal{A}(\mathcal{D}^e) - \mathcal{N}) + \mathcal{Q} = 0 \tag{22}$$

with $\mathcal{Q} = \mathcal{T}(Q)$ and $Q \in L^\infty([0\ \ T]), Q = Q' > 0$.
For any $H_1 \in L^\infty([0\ \ T]), H_1 = H_1' > 0$, the state feedback control law given by:

$$D = D^e - \mathcal{H}_1\mathcal{G}^*(X)\mathcal{P}(X - X^e) \tag{23}$$

where $\mathcal{H}_1 = \mathcal{T}(H_1)$, stabilizes globally and asymptotically the state X to X^e.

Tuning parameter: $\mathcal{H}_1$

Integral actions

- Improve the design by forwarding control [1]

$$\dot{\overline{X}} = (\mathcal{A} + \mathcal{A}(\mathcal{D}^e) - \mathcal{N})\overline{X} + \mathcal{G}(X)\overline{D}$$
$$\dot{Z} = (\mathcal{O} - \mathcal{N})Z + \mathcal{L}\mathcal{C}\overline{X} \quad (24)$$

where $\mathcal{O} = -\mathcal{O}^*$ and where $\mathcal{O}$, $\mathcal{L}$ and $\mathcal{C}$ are TB and bounded on ℓ^2.

- Primary objectives:
  - Time domain ($\int$ actions)

$$\dot{z}_1 = \ell_1(v_{\mathrm{dc}} - v_{\mathrm{dc}}^e)$$
$$\dot{z}_2 = \ell_2 i_{\mathrm{q}}$$

  - Harmonic domain

$$\begin{bmatrix}\dot{Z}_1\\ \dot{Z}_2\end{bmatrix} = \begin{bmatrix}-\mathcal{N} & 0\\ 0 & -\mathcal{N}\end{bmatrix}\begin{bmatrix}Z_1\\ Z_2\end{bmatrix} + \mathcal{L}\mathcal{C}(X - X^e) \quad (25)$$

with $\mathcal{L} = \mathrm{blkdiag}(\ell_1\mathcal{I}, \ell_2\mathcal{I}), \mathcal{C} = \begin{bmatrix}0 & \mathcal{I}\\ -\sqrt{\frac{2}{3}}\mathcal{S}_3 & 0\end{bmatrix}$ and
$\mathcal{S}_3 = \mathcal{T}(\begin{bmatrix}\sin(\omega t) & \sin(\omega t - \frac{2\pi}{3}) & \sin(\omega t + \frac{2\pi}{3})\end{bmatrix})$.

[1] [see the works of D. Astolfi, V. Andrieu, L. Praly, L. Marconi,...]

Integral Actions

- Secondary objectives:

How to cancel the k-th phasor Y_k of a given output $y(t) = C(t)x(t)$?

If $w(t) = e^{-\jmath\omega kt}y(t)$ then, for any p, $W_p = Y_{p+k}$. (k-shifted operator)
Thus, $W_0 = Y_k$

- Time domain ($\int$ of w)

$$\dot{z} = \ell e^{-\jmath\omega kt}y(t)$$

- Harmonic domain ($\equiv \int$ of Y_k)

$$\dot{Z} = -\mathcal{N}Z + \ell\mathcal{I}_k Y$$

where $\mathcal{I}_k$ denotes the $k-$shifted identity matrix.

In particular: $\dot{Z}_0 = \ell Y_k \Rightarrow \int$ of Y_k
recalling that $\mathcal{N} = diag(\jmath\omega p, p \in \mathbb{Z})$ and thus 0-line of $\mathcal{N}$ is 0.

Applied to reject 2,4,5 and 7th order phasors on i_{abc} (using higher-order Park's Transformation)

Control synthesis

Combining primary and secondary objectives into a single Z, a state feedback control is designed as follows:

- Consider the solution $\mathcal{M}$ to the harmonic Sylvester equation given by:

$$(\mathcal{O} - \mathcal{N})\mathcal{M} - \mathcal{M}(\mathcal{A} + \mathcal{A}(\mathcal{D}^e) - \mathcal{N}) + \mathcal{L}\mathcal{C} = 0 \tag{26}$$

Theorem 10 (Forwarding control)

Using $\mathcal{P}$ as provided by Theorem 9 and for any matrix functions $H_i \in L^\infty([0\ \ T])$ $H_i = H_i' > 0, i = 1,2$ such that $H_2 O - O H_2 = 0$, the state feedback control law given by:

$$D = D^e - \mathcal{H}_1 \mathcal{G}^*(X)[\mathcal{P}\bar{X} - \mathcal{M}^* \mathcal{H}_2(Z - \mathcal{M}\bar{X})] \tag{27}$$

where $\mathcal{H}_i = \mathcal{T}(H_i)$, $i = 1,2$ stabilizes globally and asymptotically the state X to X^e.

Tuning parameters: $\mathcal{H}_1$ and $\mathcal{H}_2$.
Its T-periodic time-domain counterpart is directly deduced:

$$d = d^e - H_1 G^T(x)[P(x - x^e) - M^T H_2(z - M(x - x^e))] \tag{28}$$

Stability is preserved when d is saturated ! (not shown here)

Experimental results on a bench

1. Control law is discretized with $f_s = 20kHz$
2. PLL is used to estimate electric angle and pulsation ω
3. Signal quality is assessed by calculating:
   1. For AC signals, Total Harmonic Distorsion:

$$THD_x(t) = \left(\sum_{k=2}^{k=25} \frac{|X_k(t)|^2}{|X_1(t)|^2} \right)^{\frac{1}{2}}$$

   2. For DC signals, Harmonic content:

$$HC_x(t) = \left(\sum_{k=1}^{k=25} |X_k(t)|^2 \right)^{\frac{1}{2}}$$

4. For comparison, two additional controllers are considered: PI and PI with a notch filter to attenuate unwanted harmonic behavior.

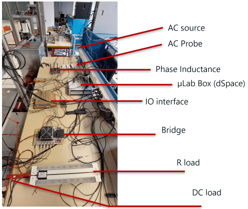
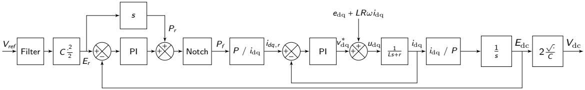


Figure: Test bench

Symbol	Quantities	Value	Unit
r	Phase resistance	1.15	Ω
L	Phase inductance	122	μH
C	Bus Capacitance	100	μF
R_L	Load nominal resistance	120	Ω
f	AC frequency	50	Hz
ω	AC pulsation	314	rad/s
E	AC rms voltage	45	V
$v_{dc,ref}$	DC load nominal voltage	150	V
$i_{dc,n}$	DC load nominal current	$\frac{v_{dc,ref}}{R_L} = 1.25$	A
i_{sink}	Programmable current load	-	A
i_{dc}	DC load actual current	$i_{sink} + \frac{v_{dc}}{R_L}$	A
$f_{in,hm}$	DC load harmonic content frequency	150	Hz

Table: Parameters Values



PI control with Notch filter

Startup V_{ref} = 150 volts, (nominal values)

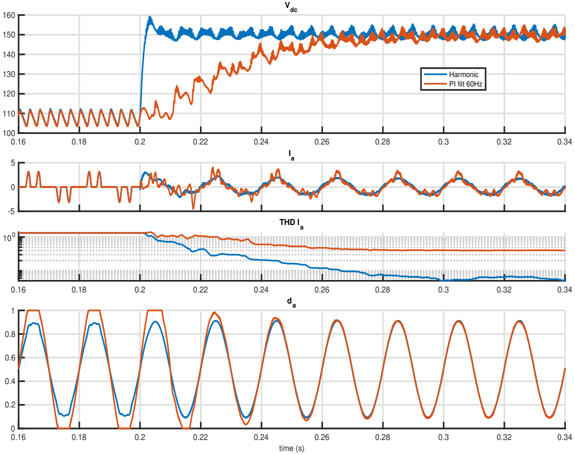


Figure: *Transcient from diode rectifier mode to controlled mode:* v_{dc}, i_a, $THDi_a$, d_a.

Load-side harmonic injection: $I_{dc} + \delta I_{dc}$ with $I_{dc} = 1\mathrm{A}$ and $\delta I_{dc} \approx 4 + 1.5\cos(3\omega t + \phi)\mathrm{A}$

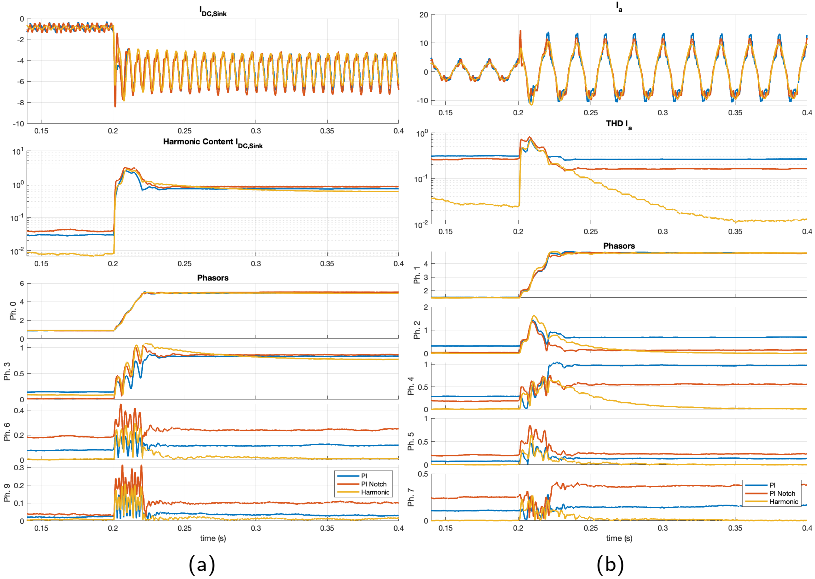
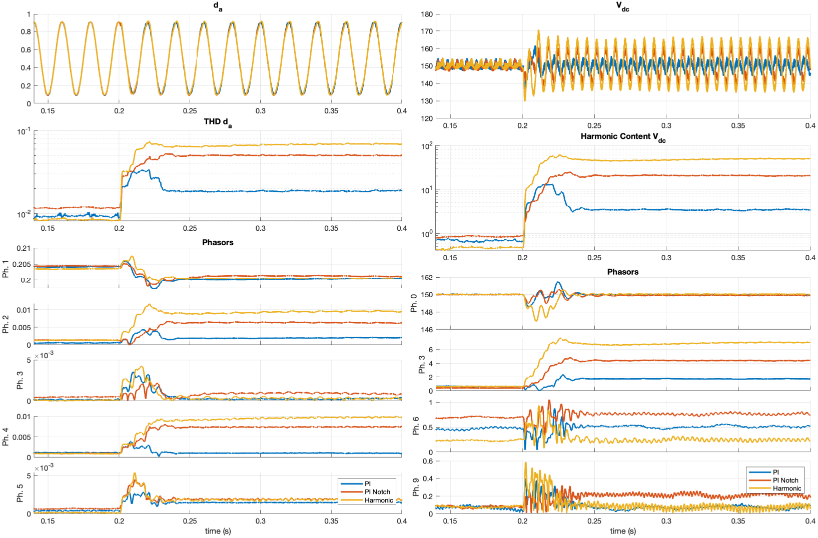


Figure: *(a) Load current I_{dc}, (b) Phase current i_a*

Load-side harmonic injection: $I_{dc} + \delta I_{dc}$ with $I_{dc} = 1\mathrm{A}$ and $\delta I_{dc} \approx 4 + 1.5\cos(3\omega t + \phi)\mathrm{A}$



(a) (b)

Figure: (a) Phase control d_A, (b) DC voltage V_{dc}

Load-side harmonic injection: $I_{dc} + \delta I_{dc}$ with $I_{dc} = 1\mathrm{A}$ and $\delta I_{dc} \approx 4 + 1.5\cos(3\omega t + \phi)\mathrm{A}$

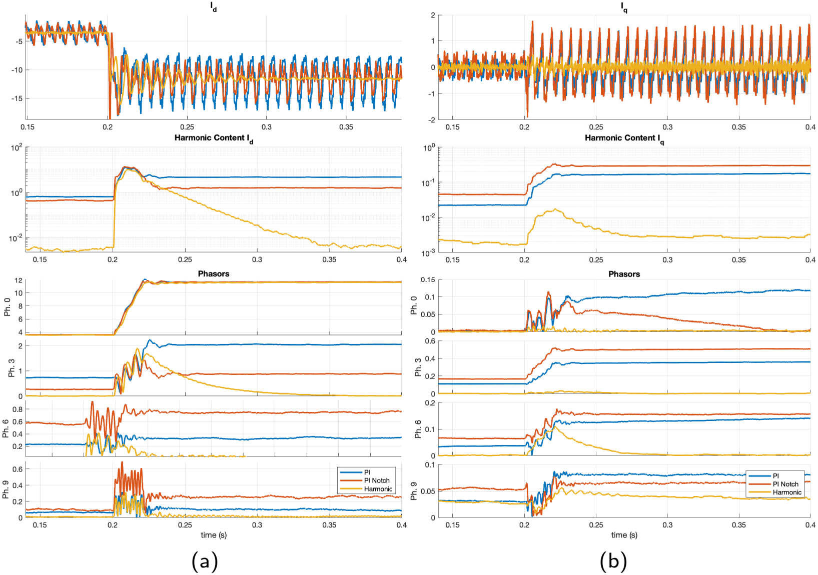


Figure: (a) Current i_d, (b) Current i_q

Optimal state feedback for LTP systems by solving TBLMIs

- Consider an unstable LTP system defined by

$$\dot{x} = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} x + \begin{pmatrix} b_{11}(t) \\ 0 \end{pmatrix} u$$

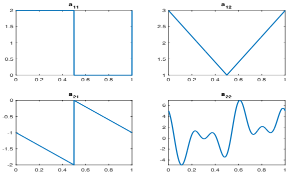


Figure: Components of A for $t \in [0\ T]$

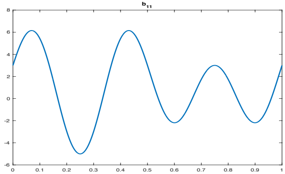


Figure: b_{11} for $t \in [0\ T]$

- The spectrum of $\mathcal{A} - \mathcal{N}$ is given by $\sigma = \{\lambda_i + j2\pi k : k \in \mathbb{Z}, i = 1, 2\}$ with $\lambda_{1,2} = \{1 \pm j1.64\}$.

- Objective: Solve the Harmonic LQR problem: $\min_U \int_0^{+\infty} X^* \mathcal{Q} X + U^* \mathcal{R} U dt$ with $\mathcal{Q} = \mathcal{T}(diag([1\ 10^4]))$ and $\mathcal{R} = \mathcal{T}(Id_m)$ and using TBLMIs.

For comparisons, 3 TBLMIs allowing to solve the LQR problem are considered:

1. *LMI*$_1$: $\mathcal{K} := \mathcal{R}^{-1}\mathcal{B}^*\mathcal{P}$

$$\max_{\mathcal{P}} tr_0(\mathcal{P}), \tag{29}$$

$$\mathcal{P} = \mathcal{P}^* > 0, \quad \begin{pmatrix} (\mathcal{A}-\mathcal{N})^*\mathcal{P} + \mathcal{P}(\mathcal{A}-\mathcal{N}) + \mathcal{Q} & \mathcal{P}\mathcal{B} \\ \mathcal{B}^*\mathcal{P} & \mathcal{R} \end{pmatrix} \geq 0$$

2. *LMI*$_2$: $\mathcal{K} := \mathcal{Y}\mathcal{S}^{-1}$

$$\min_{\mathcal{S},\mathcal{Y},\mathcal{W}} tr_0(\mathcal{W}), \quad \mathcal{S} = \mathcal{S}^* > 0 \tag{30}$$

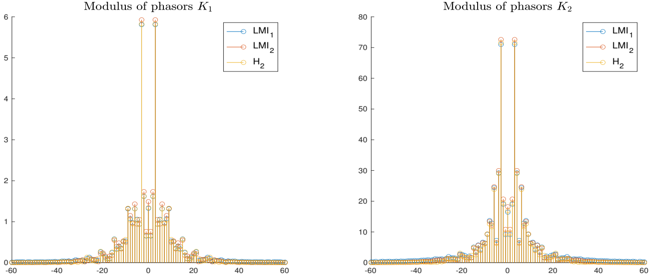
$$\begin{pmatrix} (\mathcal{A}-\mathcal{N})\mathcal{S} + \mathcal{S}(\mathcal{A}-\mathcal{N})^* + \mathcal{B}\mathcal{Y} + \mathcal{Y}^*\mathcal{B}^* & \star & \star \\ \mathcal{R}^{\frac{1}{2}}\mathcal{Y} & -\mathcal{I} & \star \\ \mathcal{Q}^{\frac{1}{2}}\mathcal{S} & 0 & -\mathcal{I} \end{pmatrix} \leq 0 \quad \begin{pmatrix} \mathcal{W} & \mathcal{I} \\ \mathcal{I} & \mathcal{S} \end{pmatrix} \geq 0$$

3. *LMI* H_2 : $\mathcal{K} := \mathcal{Y}\mathcal{S}^{-1}$

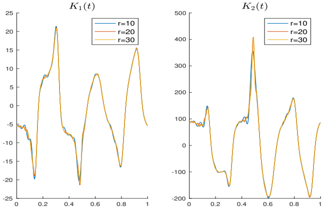
$$\min_{\mathcal{S},\mathcal{Y},\mathcal{Z}} tr_0\mathcal{Z} \quad \mathcal{S} = \mathcal{S}^* > 0$$

$$\begin{bmatrix} \mathcal{Z} & \star \\ \mathcal{S}\mathcal{C}_z^* + \mathcal{Y}^*\mathcal{D}_{zu}^* & \mathcal{S} \end{bmatrix} \geq 0 \quad \mathcal{C}_z := [\mathcal{Q}^{\frac{1}{2}}; 0], \mathcal{D}_{zu} := [0; \mathcal{R}^{\frac{1}{2}}] \tag{31}$$

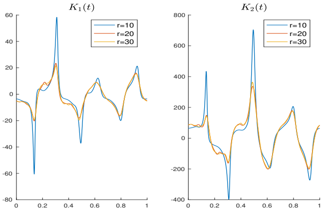
$$\begin{bmatrix} (\mathcal{A}-\mathcal{N})\mathcal{S} + \mathcal{S}(\mathcal{A}-\mathcal{N})^* + \mathcal{B}\mathcal{Y} + \mathcal{Y}^*\mathcal{B}^* + \mathcal{I} \end{bmatrix} \leq 0$$



Moduli of Phasors $K = [K_1, K_2]$ using LMI_1,LMI_2, $LMI\ H_2$ with $p = q = r = 30$

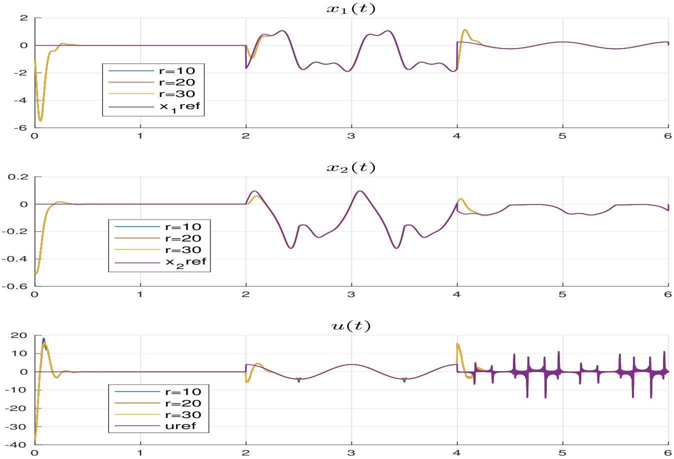


$K(t)$ over a period T for LMI_1.
$T_{comp} = 1.5, 9.5, 41s$ with $p = q = r$



$K(t)$ over a period T for $LMI\ H_2$.
$T_{comp} = 2.7,\ 22,\ 94s$ with $q = r$, $p = \frac{r}{2}$

- Under the control $u(t) = u_{ref}(t) - K(t)(x(t) - x_{ref}(t))$, the LTP system is GES on any T–periodic trajectory $(x_{ref}, u_{ref}) = \mathcal{F}^{-1}(X_{ref}, U_{ref})$ where $0 = (\mathcal{A} - \mathcal{N})X_{ref} + \mathcal{B}U_{ref}$ (harmonic equilibrium)



Conclusion

- A mathematically consistent framework for harmonic control design with dedicated tools
- Methodology: Design a control in the harmonic domain and derive its counterpart in the time domain.
- Advantages :
  - Simplified design : Periodic systems are time invariant in harmonic domain
  - Constant harmonic disturbance rejection is achieved by considering "integral actions" in harmonic domain
  - Potentially useful for electrical engineering
- But one has to cope with the infinite dimension ...

More application details in

1. Harmonic control of a three-phase rectifier bridge

   "Harmonic control of three-phase AC/DC converter" submitted to [IEEE TCST]
   available at arxiv.org/pdf/2307.06680

2. Harmonic *LQR* and Robust H_∞ and H_2 control design in

   "On solving infinite-dimensional Toeplitz Block LMIs", [CDC 2023]
   available at arxiv.org/pdf/2303.08465

   "A TBLMI Framework for Harmonic Robust Control" submitted to [IEEE TAC].
   available at arxiv.org/pdf/2311.05934

Analysis: Spectral characterization, Floquet theory

Assume $A \in L^\infty$ and $\Phi(T,0)$ is non defective (Φ is the transition matrix):

Theorem 11 (Floquet factorization revisited)

1. *The spectrum of $(\mathcal{A}-\mathcal{N})$ is an unbounded, discrete set depending on a finite number of complex values λ_i, $i = 1, \cdots, n$:*

$$\sigma := \{\lambda_i + \mathrm{j}\omega k, k \in \mathbb{Z}, i = 1, \cdots, n\}.$$

2. *The following eigenvalue decomposition takes place:*

$$(\mathcal{A}-\mathcal{N})\mathcal{V} = \mathcal{V}(\Lambda \otimes \mathcal{I} - \mathcal{N}) \tag{32}$$

with $\Lambda := diag(\lambda_i, i = 1 \cdots, n)$ and where $\mathcal{V}$ is a constant, invertible TB and bounded operator on ℓ^2.

3. *Let $V := \mathcal{T}^{-1}(\mathcal{V})$. V is an absolutely continuous, invertible and $T-$periodic matrix function and satisfies:*

$$\dot{V} = AV - V\Lambda \ a.e. \qquad V(0) := [\phi_1, \cdots, \phi_n] \tag{33}$$

where ϕ_i's are the eigenvectors of $\Phi(T,0)$ and $\lambda_i := \frac{1}{T}\log(\mu_i)$ with μ_i's the eigenvalues of $\Phi(T,0)$.

4. *If $\dot{x} = A(t)x$ (LTP system) then $z := V^{-1}x$ satisfies $\dot{z} = \Lambda z$ (LTI system)*

$\Phi(T,0)$ is easy to compute !

Proofs: "Solving Infinite-Dimensional Harmonic Lyapunov and Riccati equations", [IEEE TAC 2023]