Output Regulation for a class of linear ODE-Hyperbolic PDE-ODE systems

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Motivation

Why hyperbolic systems?

- **Conservation/balance** of scalar quantities when taking into account:
  - Evolution (e.g., **transport**) of conserved quantities in space and time
  - Finite **speed of propagation** (vs. heat equation)

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  - slow propagation speeds (e.g. traffic)
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Mathematically, this may look something like:

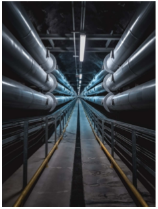
$$\partial_t \rho(t,x) = \nabla f(t,x) + S(t,x), \quad \forall (t,x) \in [0,T] \times \Omega,$$

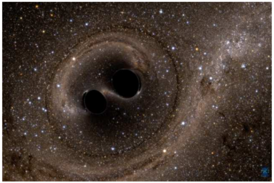
where ρ is the **quantity conserved**, f is a **flux density** and S is a **source term**.

Motivation

Many physical laws are **conservation/balance laws**, e.g. mass, charge, energy, momentum [Bastin, Coron; 2016]



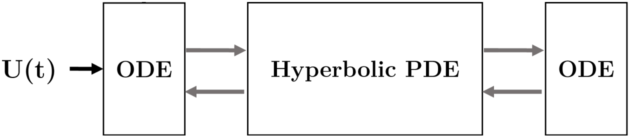


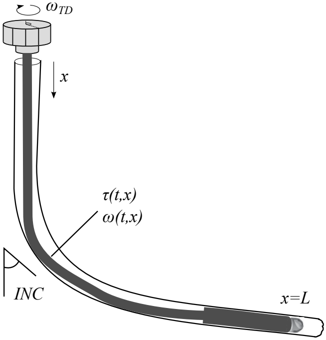
Networks of hyperbolic systems

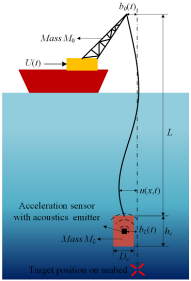
Why coupled and interconnected hyperbolic systems?

- Conservation/balance laws rarely appear isolated
  - Navier-Stokes $\rightarrow$ mass + energy + momentum
  - Propagation phenomena rarely occur in a single direction
- Systems modeled by hyperbolic PDEs do not exist in isolation, e.g.:
  - Electric transmission networks $\rightarrow$ interconnection of individual transmission lines
  - Mechanical vibrations in drilling devices $\rightarrow$ interconnection of different pipes
- Possible coupling with ODEs
  - actuator dynamics (e.g. pump, converter)
  - load dynamics (e.g. valve, motor)
  - sensor dynamics (e.g. flow-rate sensor, tachometer)



Applications: drilling systems, deepwater construction vessels [Wang et al.]





Interconnected PDE-ODE systems

- Interconnections of hyperbolic PDEs and ODEs are not a new problem.
- Many **constructive** control results based on the **backstepping approach**, e.g.:
  - Seminal paper [Krstic and Smyshlyaev, 2008]: re-interpretation of the classical Finite Spectrum Assignment [Manitius and Olbrot, 1979] (ODE + input delays)
  - Time-varying delays [Bekiaris-Liberis and Krstic, 2013, Bresch-Pietri, 2012],
  - Cascades of PDEs [Auriol et al., 2019]
  - Cascaded interconnections of hyperbolic PDE-ODE systems: [Aamo, 2012, Hasan et al., 2016, Zhou and Tang, 2012]

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- For **fully-interconnected** (non-cascaded) systems some examples include:
  - stabilizing state-feedback control law in [Di Meglio et al., 2018, Wang et al., 2018]
  - output regulation for coupled linear wave–ODE systems [Deutscher and Gabriel, 2021]

Interconnected PDE-ODE systems: control design

- For ODE-hyperbolic PDE-ODE systems with **full interconnections** (non-cascade):
  - state feedback in [Bou Saba et al., 2017] for scalar PDE system (inverible input matrix)
  - output-feedback controller based on a Byrnes-Isidori normal form for the proximal ODE, as well as a relative degree one condition in [Deutscher et al., 2018]
  - strictly-proper state-feedback control law for scalar PDE in [Bou Saba et al., 2019] requiring minimum-phase assumption (not relative degree 1)
  - extended to output-feedback control for scalar PDE in [Wang and Krstic, 2020]
  - stabilizing observer-controller robust to delays in the case of a scalar proximal ODE in [Di Meglio et al., 2020]
- Some recent results have also been obtained for interconnected PDE systems with non-linear ODEs [Irscheid et al., 2021]

Content of the presentation

What you will see in this presentation

- **Output regulation** of a general class of ODE-PDE-ODE system
  - Finite-dimensional exo-system representing the reference trajectory and disturbance dynamics.
  - Backstepping approach: integral change of coordinates
  - Time delay representation and frequency analysis
  - Stabilizing control law in the absence of the disturbance

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- A **robustification procedure**
  - Low-pass filter to make the control law strictly proper
  - Frequency analysis

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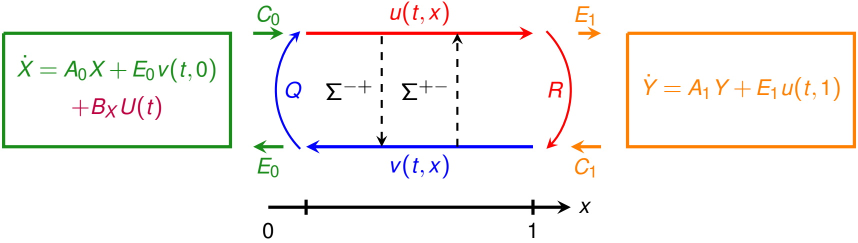
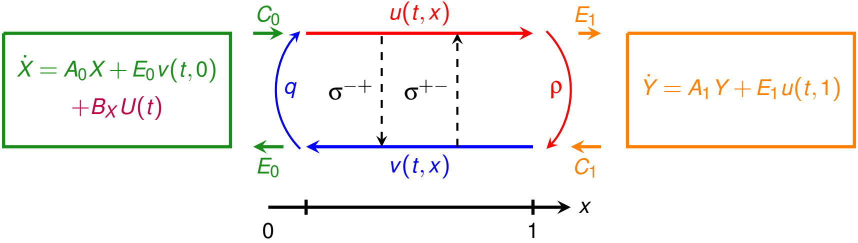
- **Observer design**
  - Backstepping approach to simplify the dynamics
  - Luenberger-like observer with tuning operators
  - Frequency analysis
  - Output-feedback control law

$$\begin{cases} \dot{X}(t) = A_0X(t) + E_0v(t,0) + B_XU(t), \\ \partial_t u(t,x) + \Lambda^+\partial_x u(t,x) = \Sigma^{++}(x)u(t,x) + \Sigma^{+-}(x)v(t,x), \\ \partial_t v(t,x) - \Lambda^-\partial_x v(t,x) = \Sigma^{-+}(x)u(t,x) + \Sigma^{--}(x)v(t,x), \\ u(t,0) = C_0X(t) + Qv(t,0), \quad v(t,1) = Ru(t,1) + C_1Y(t), \\ \dot{Y}(t) = A_{11}Y(t) + E_1u(t,1), \end{cases}$$



- **Measurement:** $y(t) = C_{\text{mes}}Y(t)$
- Same concepts for scalar and non-scalar PDEs systems

$$\begin{cases} \dot{X}(t) = A_0 X(t) + E_0 v(t,0) + B_X U(t), \\ \partial_t u(t,x) + \lambda\, \partial_x u(t,x) = \sigma^{++}(x) u(t,x) + \sigma^{+-}(x) u(t,x), \\ \partial_t v(t,x) - \mu\, \partial_x v(t,x) = \sigma^{-+}(x) u(t,x) + \sigma^{--}(x) v(t,x), \\ u(t,0) = C_0 X(t) + q v(t,0), \quad v(t,1) = \rho u(t,1) + C_1 Y(t), \\ \dot{Y}(t) = A_1 Y(t) + E_1 u(t,1), \end{cases}$$



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System under consideration: ODE-PDE-ODE

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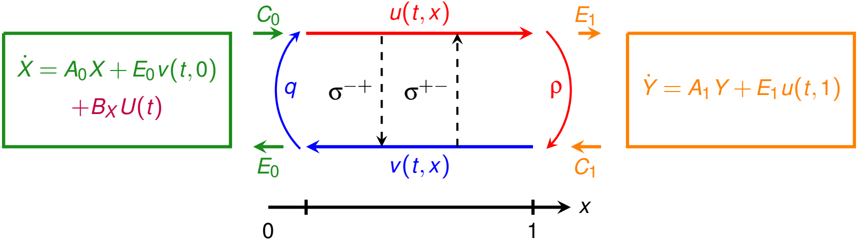

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- Initial conditions in H^1 with appropriate compatibility conditions $\rightarrow$ **well-posedness**

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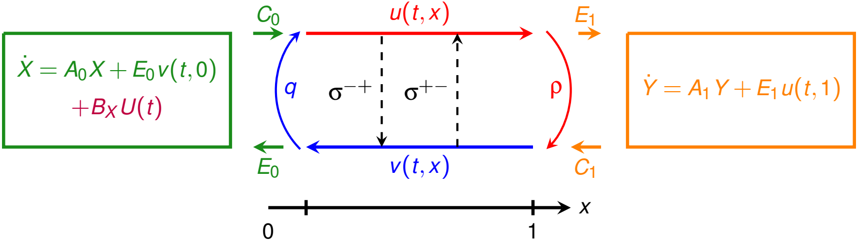

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- Same concepts for scalar and non-scalar PDEs systems
- Diagonal terms can be removed with exp. change of coordinates
- Initial conditions in H^1 with appropriate compatibility conditions $\to$ **well-posedness**
- Stabilization in the sense of the L^2-norm

System under consideration: well-posedness and stabilization objective

$$
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Well-posedness in open-loop

For every initial condition $(X_0, u_0, v_0, Y_0) \in \mathbb{R}^p \times H^1([0,1],\mathbb{R}^2) \times \mathbb{R}^q$ that verifies the compatibility conditions

$$u_0(0) = C_0 X(t) + Q v_0(0), \quad v_0(1) = R u_0(1) + C_1 Y(t)$$

there exists one and one only (X, u, v, Y) which is a **solution to the open-loop Cauchy problem** (i.e., $U \equiv 0$).
Moreover, there exists $\kappa_0 > 0$ such that for every $(X_0, u_0, v_0, Y_0) \in \mathbb{R}^p \times H^1([0,1],\mathbb{R}^2) \times \mathbb{R}^q$ satisfying the compatibility conditions, the unique solution verifies

$$||(X(t), u(t,\cdot), v(t,\cdot), Y(t))||_\chi \leq \kappa_0 \mathrm{e}^{\kappa_0 t} ||(X_0, u_0, v_0, Y_0)||_\chi, \quad \forall t \in [0,\infty).$$

where $||(X(t), u(t,\cdot), v(t,\cdot), Y(t))||_\chi = \sqrt{||X(t)||^2_{\mathbb{R}^p} + ||u(t,\cdot)||^2_{L^2} + ||v(t,\cdot)||^2_{L^2} + ||Y(t)||^2_{\mathbb{R}^q}}.$

System under consideration: well-posedness and stabilization objective

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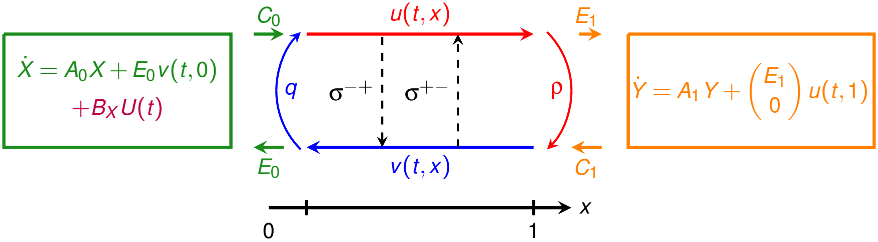
Stabilization objective

Design a continuous control input that **exponentially stabilizes** the system in the sense of the L^2-norm, i.e. there exist κ_0 and $\nu > 0$ such that for any initial condition $(X_0, u_0, v_0, Y_0) \in \mathbb{R}^p \times H^1([0,1], \mathbb{R}^2) \times \mathbb{R}^q$, we have

$$||(X(t), u(t,\cdot), v(t,\cdot), Y(t))||_\chi \leq \kappa_0 \mathrm{e}^{-\nu t} ||(X_0, u_0, v_0, Y_0)||_\chi, \; 0 \leq t$$

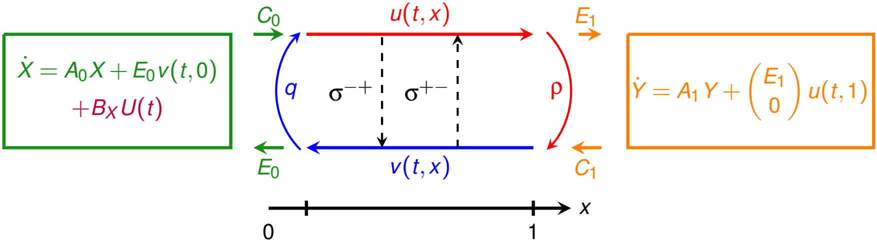


<u>**Augmented variable:**</u> $Y(t) = \left(Y_1^\top(t)\ ,\ Y_2^\top(t)\right)^\top$

- Y_1 is the "real" ODE state
- Y_2 is an **exogenous input**: **disturbance** Y_{dist} and/or a **reference trajectory** Y_{ref}

$$\dot Y(t) = A_1 Y(t) + \begin{pmatrix} E_1 \\ 0_{q_2 \times 1} \end{pmatrix} u(t,1), \text{ with } A_1 = \begin{pmatrix} A_{11} & A_{12} \\ 0_{q_2 \times q_1} & A_{22} \end{pmatrix},$$

Output-regulation problem



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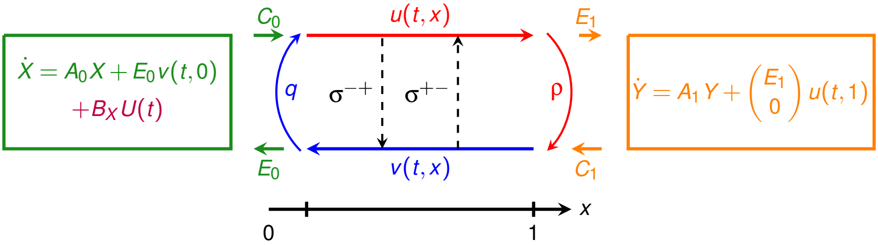
Virtual output: $\varepsilon(t) = C_e Y(t) = \begin{pmatrix} C_{e1} & C_{e2} \end{pmatrix} Y(t)$

Control objective

Design a control law $U(t)$ s.t. **the virtual output** $\varepsilon(t)$ exp. converges to zero.



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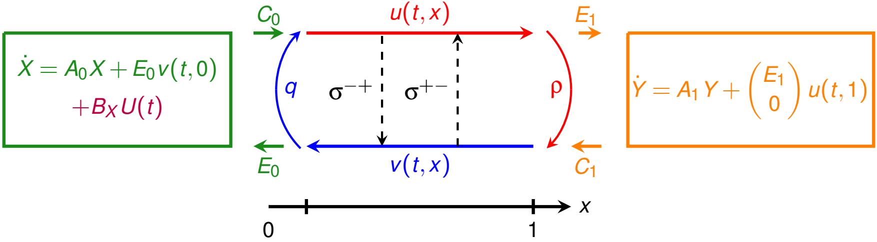
- **Output regulation problem:** $C_{e1} \not\equiv 0$, and $C_{e2} \equiv 0$: we want to regulate to zero a linear combination of components of $Y_1(t)$ in the presence of a disturbance $Y_2(t)$.



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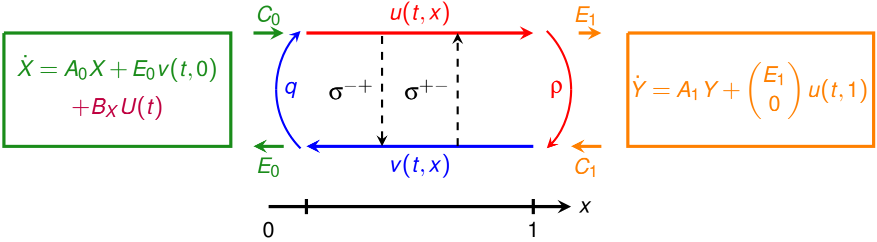
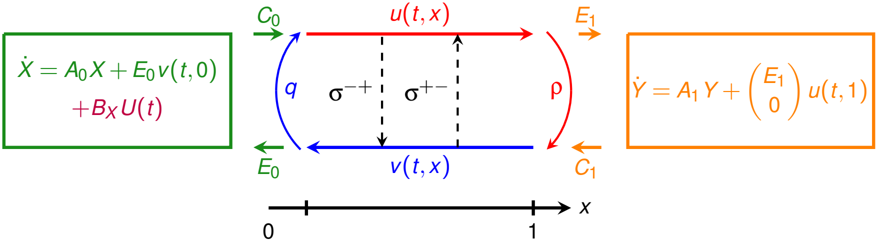
- **Output tracking problem:** $C_{e1,i} - C_{e2,j} = 0$, (other components = 0): we want the i^{th} component of the output Y_1 to converge towards the j^{th} component of a known trajectory Y_2.



Assumption 1: Stabilizability

The pairs (A_0, B_0) and (A_{11}, E_1) are **stabilizable**, i.e. there exist $F_0 \in \mathbb{R}^{r \times p}$, $F_1 \in \mathbb{R}^{n \times q_1}$ such that $\bar{A}_0 \doteq A_0 + B_X F_0$ and $\bar{A}_{11} \doteq A_{11} + E_1 F_1$ are Hurwitz.

- Classical requirement found in most of the papers dealing with ODE-PDE-ODE
- Not overly conservative (necessary to stabilize Y, slightly conservative for X).



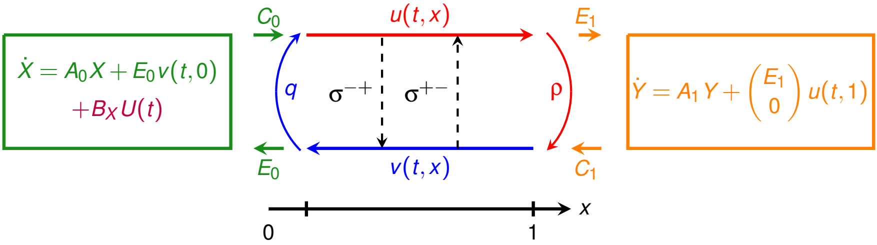
Assumption 2

For all $s \in \mathbb{C}_0$, the matrices (A_0, B_X, C_0) satisfy

$$\operatorname{rank} \begin{pmatrix} s\mathrm{Id} - A_0 & B_X \\ C_0 & 0_{n \times r} \end{pmatrix} = p + 1 = p + n.$$

- The function $P_0(s) = C_0(s\mathrm{Id} - \bar{A}_0)^{-1} B_X$ does not have any zeros in $\mathbb{C}^+$
- **Stable right inverse** of $P_0(s)$

Structural assumptions

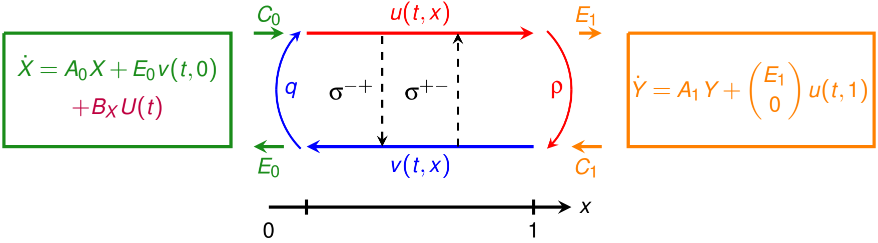


Assumption 3: Delay-robustness

The coefficients ρ and q verifiy $|\rho q| < 1$.

- No asymptotic chain of eigenvalues with non-negative real parts
- Necessary for (delay-) robust stabilization

Structural assumptions



Assumption 4: detectability

The pairs (A_1, C), (A_0, C_0) **are detectable** (i.e. there exist $L_0 \in \mathbb{R}^{p\times n}$ and $L_1 \in \mathbb{R}^{q\times d}$ such that $\tilde{A}_1 \doteq A_1 + L_1 C_{mes}$ and $\tilde{A}_0 \doteq A_0 + L_0 C_0$ are Hurwitz).

- Classical requirement found in most of the papers dealing with ODE-PDE-ODE
- Not overly conservative (necessary for reconstruction of X_0, slightly conservative for Y).

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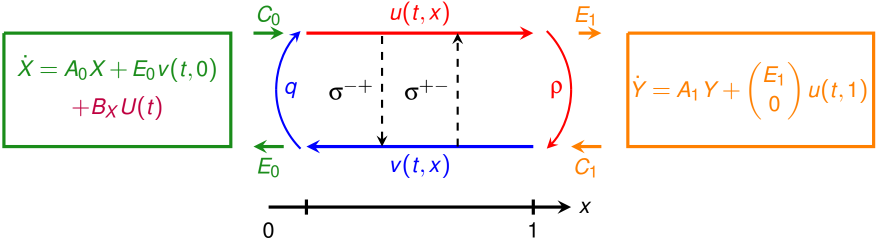


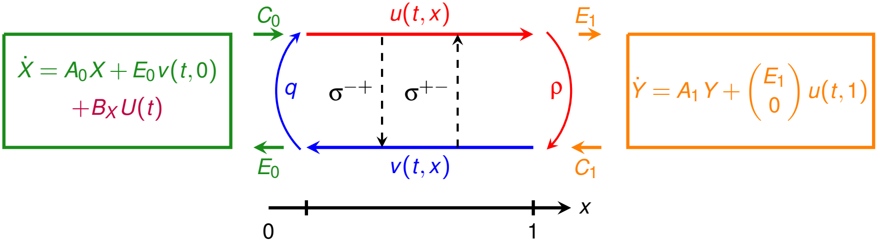
Assumption 5

For all $s \in \mathbb{C}^+$, the matrices (A_1, E_1, C) satisfy

$$\text{rank}\left(\begin{pmatrix} s\text{Id} - A_1 & E_1 \\ C_{mes} & 0 \end{pmatrix}\right) = q + 1 = q + n. \tag{1}$$

- Necessary to independently reconstruct the different PDE boundary values by inverting the Y dynamics.
- The function $P_1(s) \doteq C_{mes}(s\text{Id} - \tilde{A}_1)^{-1} E_1$ does not have any zeros in $\mathbb{C}^+$
- **Stable left-inverse** of $P_1(s)$

Structural assumptions



Assumption 6

The matrix A_{22} is marginally stable, i.e., all its eigenvalues have zero real parts. There exist matrices $T_a \in \mathbb{R}^{q_1 \times q_2}, F_a \in \mathbb{R}^{n \times q_2}$ solutions to the **regulator equations**:

$$\begin{cases} -A_{11} T_a + T_a A_{22} + A_{12} = -E_1 F_a, \\ -C_{e1} T_a + C_{e2} = 0. \end{cases}$$

- Non-resonance condition.
- A_{11} and A_{22} have disjoint spectra, and the number of outputs we regulate is coherent with the number of inputs.
- The matrices T_a, F_a can be computed using a Schur triangulation.

Control design: strategy.

- Backstepping transformation to simplify the dynamics and the design of the control law.
- The regulation problem rewrites as a stabilization problem.
- Time-delay representation and frequency analysis.
- Low-pass filtering of the control law to make it **strictly proper**.

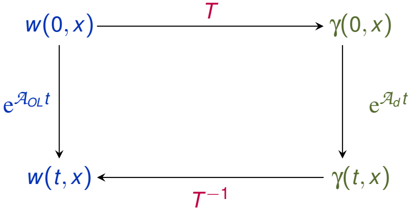
Backstepping methodology

- Map the original system to a *target system* for which the stability analysis is easier.
- Variable change: integral transformation, classically Volterra transform of the *second kind*

$$\alpha(t,x) = u(t,x) - \int_0^x K^{uu}(x,\xi)u(t,\xi) + K^{uv}(x,\xi)v(t,\xi)d\xi,$$

$$\beta(t,x) = v(t,x) - \int_0^x K^{vu}(x,\xi)u(t,\xi) + K^{vv}(x,\xi)v(t,\xi)d\xi,$$

Condensed form: $\gamma(t,x) = w(t,x) - \int_0^x K(x,y)w(t,y)dy.$

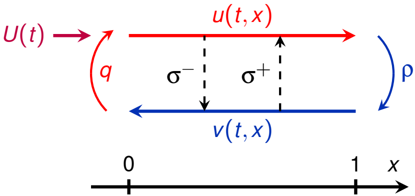


Limitations

- Choice of an adequate target system.
- Proof of existence and invertibility of an adequate backstepping transform.

Objective: Move the in-domain coupling terms at the actuated boundary.

$$u_t(t,x) + \lambda u_x(t,x) = \sigma^+ v(t,x),$$
$$v_t(t,x) - \mu v_x(t,x) = \sigma^- u(t,x).$$



$u(t,0) = qv(t,0) + U(t)$

$v(t,1) = \rho u(t,1)$

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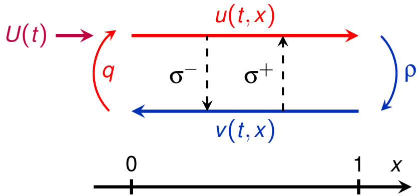
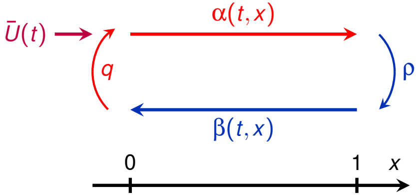


$u(t,0) = qv(t,0) + U(t)$

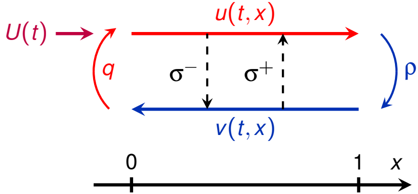
$v(t,1) = \rho u(t,1)$

$$\alpha_t(t,x) + \lambda \alpha_x(t,x) = 0,$$
$$\beta_t(t,x) - \mu \beta_x(t,x) = 0.$$



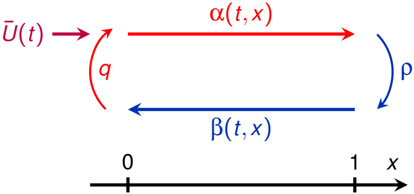
Objective: Move the in-domain coupling terms at the actuated boundary.

$$u_t(t,x)+\lambda u_x(t,x)=\sigma^{+}v(t,x),$$
$$v_t(t,x)-\mu v_x(t,x)=\sigma^{-}u(t,x).$$



$$u(t,0)=qv(t,0)+U(t)$$
$$v(t,1)=\rho u(t,1)$$

$$\alpha_t(t,x)+\lambda\alpha_x(t,x)=0,$$
$$\beta_t(t,x)-\mu\beta_x(t,x)=0.$$



$$\alpha(t,0)=q\beta(t,0)+U(t)$$
$$-\int_0^1 N^{\alpha}(\xi)\alpha(t,\xi)+N^{\beta}(\xi)\beta(t,\xi)d\xi.$$
$$\beta(t,1)=\rho\alpha(t,1)$$

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$$u_t(t,x) + \lambda u_x(t,x) = \sigma^+ v(t,x),$$
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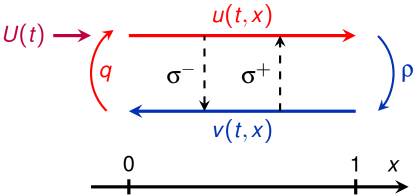

$$u(t,0) = qv(t,0) + U(t)$$
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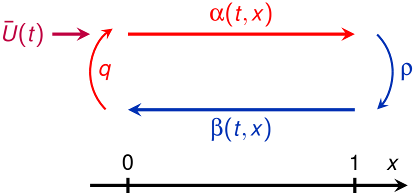


$$\alpha(t,0) = q\beta(t,0) + U(t) - \int_0^1 N^\alpha(\xi)\alpha(t,\xi) + N^\beta(\xi)\beta(t,\xi)d\xi.$$
$$\beta(t,1) = \rho\alpha(t,1)$$

Natural control law

$$U(t) = -q\beta(t,0) + \int_0^1 \left(N^\alpha(\xi)\alpha(t,\xi) + N^\beta(\xi)\beta(t,\xi)\right) d\xi.$$

$$X(t)=\xi(t)+\int_0^1 M^{12}(y)\alpha(t,y)+M^{13}(y)\beta(t,y)\mathrm{d}y+\begin{bmatrix}M^{14} & M^{15}\end{bmatrix}\eta(t),$$
$$u(t,x)=\alpha(t,x)+\int_x^1 M^{22}(x,y)\alpha(y)+M^{23}(x,y)\beta(y)\mathrm{d}y+\begin{bmatrix}M^{24}(x) & M^{25}(x)\end{bmatrix}\eta(t),$$
$$v(t,x)=\beta(t,x)+\int_x^1 M^{32}(x,y)\alpha(y)+M^{33}(x,y)\beta(y)\mathrm{d}y+\begin{bmatrix}M^{34}(x) & M^{35}(x)\end{bmatrix}\eta(t),$$
$$Y(t)=\eta(t).$$

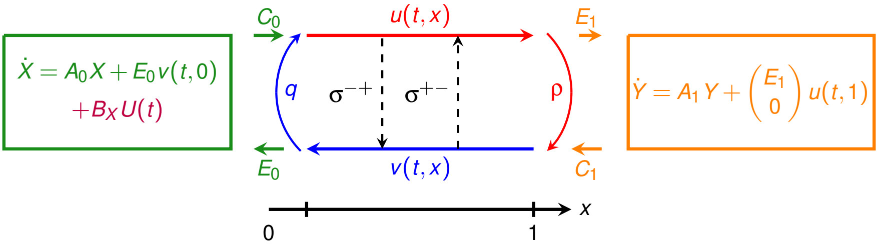
- Triangular transformation: invertible.

$$\begin{pmatrix}X(t)\\u(t,x)\\v(t,x)\\Y(t)\end{pmatrix}=\begin{pmatrix}\mathbf{Id} & \int_0^1 M^{12}(y)dy & \int_0^1 M^{13}(y)dy & [M^{14}\ M^{15}]\\ \mathbf{0} & \mathbf{Id}+\int_x^1 M^{22}(x,y)dy & \int_x^1 M^{23}(x,y)dy & [M^{24}(x)\ M^{25}(x)]\\ \mathbf{0} & \int_x^1 M^{32}(x,y)dy & \mathbf{Id}+\int_x^1 M^{33}(x,y)dy & [M^{34}(x)\ M^{35}(x)]\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Id}\end{pmatrix}\begin{pmatrix}\xi(t)\\\alpha(t,x)\\\beta(t,x)\\\eta(t)\end{pmatrix}$$

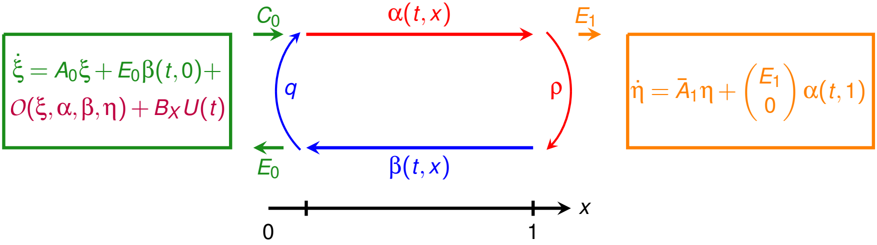
- Kernels are bounded functions.
- Unique solution due to the rank condition on C_0.

Backstepping: Target system

Original system:



Target system:



Backstepping: Target system

Original system:

$$\begin{cases} \dot{X}(t) = A_0 X(t) + E_0 v(t,0) + B_X U(t), \\ \partial_t u(t,x) + \Lambda^+ \partial_x u(t,x) = \sigma^{+-}(x) u(t,x), \\ \partial_t v(t,x) - \Lambda^- \partial_x v(t,x) = \sigma^{-+}(x) u(t,x), \\ u(t,0) = C_0 X(t) + q v(t,0), \quad v(t,1) = \rho u(t,1) + C_1 Y(t), \\ \dot{Y}(t) = A_1 Y(t) + \begin{pmatrix} E_1 & 0 \end{pmatrix}^\top u(t,1), \end{cases}$$

Target system:

$$\begin{cases} \dot{\xi}(t) = \bar{A}_0 \xi(t) + \bar{E}_1 \alpha(t,1) + \bar{E}_0 \beta(t,0) + M\eta(t) \\ \qquad + \int_0^1 M_\alpha(y)\alpha(t,y) + M_\beta(y)\beta(t,y) dy + B_X \bar{U}(t), \\ \partial_t \alpha(t,x) + \Lambda^+ \partial_x \alpha(t,x) = 0, \\ \partial_t \beta(t,x) - \Lambda^- \partial_x \beta(t,x) = 0, \\ \alpha(t,0) = C_0 \xi(t) + q\beta(t,0), \quad \beta(t,1) = \rho\alpha(t,1), \\ \dot{\eta}(t) = \bar{A}_1 \eta(t) + \begin{pmatrix} E_1 & 0 \end{pmatrix}^\top \alpha(t,1), \end{cases}$$

$$\bar{A}_0 = A_0 + B_X F_0, \ \bar{A}_1 = \begin{pmatrix} A_{11} + E_1 F_1 & A_{12} + E_1 (F_a + F_1 T_a) \\ 0 & A_{22} \end{pmatrix}$$

Advantages of the target system:

- Simplified in-domain couplings.
- Almost a "cascade structure"
- To stabilize the whole system, we can focus on the stabilization of ξ.

A cascade structure

$$\begin{cases} \dot{\xi}(t) = \bar{A}_0\xi(t) + \bar{E}_1\alpha(t,1) + \bar{E}_0\beta(t,0) + M\eta(t) \\ \qquad + \int_0^1 M_\alpha(y)\alpha(t,y) + M_\beta(y)\beta(t,y)dy + B_X\bar{U}(t), \\ \partial_t\alpha(t,x) + \Lambda^+\partial_x\alpha(t,x) = 0, \\ \partial_t\beta(t,x) - \Lambda^-\partial_x\beta(t,x) = 0, \\ \alpha(t,0) = C_0\xi(t) + q\beta(t,0), \quad \beta(t,1) = \rho\alpha(t,1), \\ \dot{\eta}(t) = \bar{A}_1\eta(t) + \begin{pmatrix} E_1 & 0 \end{pmatrix}^\top \alpha(t,1), \end{cases}$$

Stability and regulation

If $C_0\xi$ exp. converges to zero, then $\varepsilon(t) \to 0$. Furthermore, the trajectories are bounded.

A cascade structure

Assumption 6

The matrix A_{22} is marginally stable, i.e., all its eigenvalues have zero real parts. There exist matrices $T_a \in \mathbb{R}^{q_1 \times q_2}, F_a \in \mathbb{R}^{n \times q_2}$ solutions to the **regulator equations**:

$$\begin{cases} -A_{11}T_a + T_a A_{22} + A_{12} = -E_1 F_a, \\ -C_{e1}T_a + C_{e2} = 0. \end{cases}$$

Stability and regulation

If ξ exp. converges to zero, then $\varepsilon(t) \to 0$. Furthermore, the trajectories are bounded.

Proof: If $C_0\xi$ converges to zero, then so does $||(\alpha, \beta)||_{L^2}$.

- We have

$$\begin{aligned} \dot{Y}_1 &= (A_{11} + E_1 F_1)Y_1(t) + (A_{12} + E_1(F_a + F_1 T_a))Y_2(t) + E_1\alpha(t,1) \\ &= (A_{11} + E_1 F_1)Y_1(t) + (A_{11}T_a - E_1 F_a - T_a A_{22})Y_2(t) + E_1(F_a + F_1 T_a)Y_2(t) + E_1\alpha(t,1), \end{aligned}$$

$$\Rightarrow \overbrace{(Y_1 + T_a Y_2)}^{\cdot}(t) = \bar{A}_{11}(Y_1 + T_a Y_2) + \overbrace{E_1\alpha(t,1)}^{\longrightarrow 0}.$$

A cascade structure

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- We have

$$\dot{Y}_1 = (A_{11} + E_1F_1)Y_1(t) + (A_{12} + E_1(F_a + F_1T_a))Y_2(t) + E_1\alpha(t,1)$$
$$= (A_{11} + E_1F_1)Y_1(t) + (A_{11}T_a - E_1F_a - T_aA_{22})Y_2(t) + E_1(F_a + F_1T_a)Y_2(t) + E_1\alpha(t,1),$$
$$\Rightarrow \overbrace{(Y_1 + T_aY_2)}^{\cdot}(t) = \bar{A}_{11}(Y_1 + T_aY_2) + \overbrace{E_1\alpha(t,1)}^{\longrightarrow 0}.$$

- $Y_1 + T_aY_2$ exp. stable $\Rightarrow$ $C_e(Y_1 + T_aY_2)(t) = C_{e1}Y_1(t) + C_{e2}Y_2(t) = \varepsilon(t)$ goes to zero.

Assumption 6

The matrix A_{22} is marginally stable, i.e., all its eigenvalues have zero real parts. There exist matrices $T_a \in \mathbb{R}^{q_1 \times q_2}, F_a \in \mathbb{R}^{n \times q_2}$ solutions to the **regulator equations**:

$$\begin{cases} -A_{11} T_a + T_a A_{22} + A_{12} = -E_1 F_a, \\ -C_{e1} T_a + C_{e2} = 0. \end{cases}$$

Stability and regulation

If ξ exp. converges to zero, then $\varepsilon(t) \to 0$. Furthermore, the trajectories are bounded.

Proof: If $C_0 \xi$ converges to zero, then so does $||(\alpha, \beta)||_{L^2}$.

- We have

$$\begin{aligned} \dot{Y}_1 &= (A_{11} + E_1 F_1) Y_1(t) + (A_{12} + E_1(F_a + F_1 T_a)) Y_2(t) + E_1 \alpha(t,1) \\ &= (A_{11} + E_1 F_1) Y_1(t) + (A_{11} T_a - E_1 F_a - T_a A_{22}) Y_2(t) + E_1(F_a + F_1 T_a) Y_2(t) + E_1 \alpha(t,1), \end{aligned}$$

$$\Rightarrow \overbrace{(Y_1 + T_a Y_2)}^{\cdot}(t) = \bar{A}_{11}(Y_1 + T_a Y_2) + \overbrace{E_1 \alpha(t,1)}^{\longrightarrow 0}.$$

- $Y_1 + T_a Y_2$ exp. stable $\Rightarrow$ $C_e(Y_1 + T_a Y_2)(t) = C_{e1} Y_1(t) + C_{e2} Y_2(t) = \varepsilon(t)$ goes to zero.
- Invertibility + boundedness of the backstepping transf. implies boundedness of the state.

$$\begin{aligned}\alpha_t(t,x)+\lambda\alpha_x(t,x) &= 0\\ \beta_t(t,x)-\mu\beta_x(t,x) &= 0\end{aligned}$$

$$\begin{aligned}\alpha(t,0) &= q\beta(t,0)+C_0\xi(t)\\ \beta(t,1) &= \rho\alpha(t,1)\end{aligned}$$

Time-delay representation

$$\alpha_t(t,x) + \lambda\alpha_x(t,x) = 0 \rightarrow \text{Transport equation}$$
$$\beta_t(t,x) - \mu\beta_x(t,x) = 0 \rightarrow \text{Transport equation}$$

$$\alpha(t,0) = q\beta(t,0) + C_0\xi(t)$$
$$\beta(t,1) = \rho\alpha(t,1)$$

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$$\alpha(t,0) = q\beta(t,0) + C_0\xi(t)$$
$$\beta(t,1) = \rho\alpha(t,1)$$

Method of characteristics:

$$\alpha(t,x) = \alpha(t - \frac{x}{\lambda}, 0), \quad \beta(t,x) = \rho\alpha(t - \frac{(1-x)}{\mu} - \frac{1}{\lambda}, 0)$$

Difference Equation satisfied by $\alpha(t,0)$

$$\alpha(t,0) = \rho q\alpha(t-\tau,0) + C_0\xi(t), \quad t > \frac{1}{\lambda} + \frac{1}{\mu} = \tau$$

Using the Laplace transform: $(1 - \rho q e^{-\tau s})\alpha(s,0) = C_0\xi(s)$

We can kill the α and β terms to obtain ξ-terms!

Time-delay representation

$$\dot{\eta}(t) = \bar{A}_1\eta(t) + \begin{pmatrix} E_1 & 0 \end{pmatrix}^\top \alpha(t,1)$$
$$\dot{\xi}(t) = \bar{A}_0\xi(t) + \bar{E}_1\alpha(t,1) + \bar{E}_0\beta(t,0) + M\eta(t) + \int_0^1 M_\alpha(y)\alpha(t,y) + M_\beta(y)\beta(t,y)dy + B_X\bar{U}(t).$$

Laplace transform on η_1

$$\eta_1(s) = (s\mathrm{Id} - \bar{A}_{11})^{-1}(\bar{A}_{12}\eta_2(s) + E_1 e^{-\frac{s}{\lambda}}\alpha(s,0))$$

We can get rid of the η_1-terms!

Time-delay representation

$$\dot{\eta}(t) = \bar{A}_1\eta(t) + \begin{pmatrix} E_1 & 0 \end{pmatrix}^\top \alpha(t,1)$$
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We can get rid of the η_1-terms!

Laplace transform on ξ

$$(s\mathrm{Id} - \bar{A}_0)\xi(s) = G(s)C_0\xi(s) + H(s)\eta_2(s) + B_X\bar{U}(s),$$

$P_0 = C_0(s\mathrm{Id} - \bar{A}_0)^{-1}B_X$ admits a stable right inverse P_0^+.

$$C_0\xi(s) = C_0(s\mathrm{Id} - \bar{A}_0)^{-1}G(s)C_0\xi(s) + C_0(s\mathrm{Id} - \bar{A}_0)^{-1}H(s)\eta_2(s) + P_0(s)\bar{U}(s),$$

Time-delay representation

$$\dot{\eta}(t) = \bar{A}_1\eta(t) + \begin{pmatrix} E_1 & 0 \end{pmatrix}^\top \alpha(t,1)$$
$$\dot{\xi}(t) = \bar{A}_0\xi(t) + \bar{E}_1\alpha(t,1) + \bar{E}_0\beta(t,0) + M\eta(t) + \int_0^1 M_\alpha(y)\alpha(t,y) + M_\beta(y)\beta(t,y)dy + B_X\bar{U}(t).$$

Laplace transform on η_1

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We can get rid of the η_1-terms!

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$$C_0\xi(s) = C_0(s\mathrm{Id} - \bar{A}_0)^{-1}G(s)C_0\xi(s) + C_0(s\mathrm{Id} - \bar{A}_0)^{-1}H(s)\eta_2(s) + P_0(s)\bar{U}(s),$$

Stabilizing control law

$$\bar{U}(s) = \underbrace{-P_0^+(s)C_0(s\mathrm{Id} - \bar{A}_0)^{-1}G(s)C_0\xi(s)}_{\text{stabilization}} \quad \underbrace{-P_0^+(s)C_0(s\mathrm{Id} - \bar{A}_0)^{-1}H(s)\eta_2(s)}_{\text{disturbance rejection or tracking}}$$

A non strictly proper control law

Stabilizing control law

$$\bar{U}(s) = \underbrace{-P_0^+(s)C_0(s\mathrm{Id}-\bar{A}_0)^{-1}G(s)C_0\xi(s)}_{\text{stabilization}} \quad \underbrace{-P_0^+(s)C_0(s\mathrm{Id}-\bar{A}_0)^{-1}H(s)\eta_2(s)}_{\text{disturbance rejection or tracking}}$$

$$= F_\xi(s)\xi(s) + F_\eta(s)\eta_2(s)$$

- The control law ay not be **strictly proper** due to $P_0^+(s) \rightarrow$ Robustness issues.
- We can make $F_\eta(s)$ strictly proper using our prior knowledge of the dynamics.
- We can make $F_\xi(s)$ strictly proper using a low-pass filter.

Filtering of the control input

$$F_\xi(s) = -P_0^+(s)C_0(s\mathrm{Id} - \bar{A}_0)^{-1}G(s)C_0, \quad F_\eta(s) = -P_0^+(s)C_0(s\mathrm{Id} - \bar{A}_0)^{-1}H(s)$$

Filtered control law

Let $\mathrm{w}(s)$ be any low-pass filter, with a sufficiently high relative degree, and $0 < \delta < 1$ such that

$$\forall x \in \mathbb{R},\ |1 - \mathrm{w}(jx)| \leq \frac{1-\delta}{\|G\|_\infty \bar{\sigma}(C_0(jx\mathrm{Id} - \bar{A}_0)^{-1})},$$

then $\bar{U}(s) = w(s)F_\xi(s)\xi(s) + \bar{F}_\eta(s)\eta_2(s)$ stabilizes $C_0\xi(s)$

Proof: Let $\Phi(s) = (1 - w(s))C_0(s\mathrm{Id} - \bar{A}_0)^{-1}G(s)$.

Filtering of the control input

$$F_\xi(s) = -P_0^+(s)C_0(s\mathrm{Id} - \bar{A}_0)^{-1}G(s)C_0, \quad F_\eta(s) = -P_0^+(s)C_0(s\mathrm{Id} - \bar{A}_0)^{-1}H(s)$$

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Proof: Let $\Phi(s) = (1 - w(s))C_0(s\mathrm{Id} - \bar{A}_0)^{-1}G(s)$.

- Φ is stable and strictly proper

Filtering of the control input

$$F_\xi(s) = -P_0^+(s) C_0 (s\mathrm{Id} - \bar{A}_0)^{-1} G(s) C_0, \quad F_\eta(s) = -P_0^+(s) C_0 (s\mathrm{Id} - \bar{A}_0)^{-1} H(s)$$

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then $\bar{U}(s) = w(s) F_\xi(s) \xi(s) + \bar{F}_\eta(s) \eta_2(s)$ stabilizes $C_0 \xi(s)$

Proof: Let $\Phi(s) = (1 - w(s)) C_0 (s\mathrm{Id} - \bar{A}_0)^{-1} G(s)$.

- Φ is stable and strictly proper
- $G(s)$ is unif. bounded, we have $\bar{\sigma}(G(jx)) \leq \|G\|_\infty$ for all x

Filtering of the control input

$$F_\xi(s) = -P_0^+(s)C_0(s\mathrm{Id} - \bar{A}_0)^{-1}G(s)C_0, \quad F_\eta(s) = -P_0^+(s)C_0(s\mathrm{Id} - \bar{A}_0)^{-1}H(s)$$

Filtered control law

Let $\mathrm{w}(s)$ be any low-pass filter, with a sufficiently high relative degree, and $0 < \delta < 1$ such that

$$\forall x \in \mathbb{R},\ |1 - \mathrm{w}(jx)| \le \frac{1-\delta}{\|G\|_\infty \bar{\sigma}(C_0(jx\mathrm{Id} - \bar{A}_0)^{-1})},$$

then $\bar{U}(s) = w(s)F_\xi(s)\xi(s) + \bar{F}_\eta(s)\eta_2(s)$ stabilizes $C_0\xi(s)$

Proof: Let $\Phi(s) = (1 - w(s))C_0(s\mathrm{Id} - \bar{A}_0)^{-1}G(s)$.

- Φ is stable and strictly proper
- $G(s)$ is unif. bounded, we have $\bar{\sigma}(G(jx)) \le \|G\|_\infty$ for all x
- We have $\bar{\sigma}(\phi(jx)) \le 1 - \delta \Rightarrow ||\Phi||_\infty < 1$

Filtering of the control input

$$F_\xi(s) = -P_0^+(s)C_0(s\mathrm{Id} - \bar{A}_0)^{-1}G(s)C_0, \quad F_\eta(s) = -P_0^+(s)C_0(s\mathrm{Id} - \bar{A}_0)^{-1}H(s)$$

Filtered control law

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- Characteristic equation $(1 - \Phi(s))C_0\xi(s) = 0 \to$ **exponential stability**

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- Characteristic equation $(1 - \Phi(s))C_0\xi(s) = 0 \rightarrow$ **exponential stability**

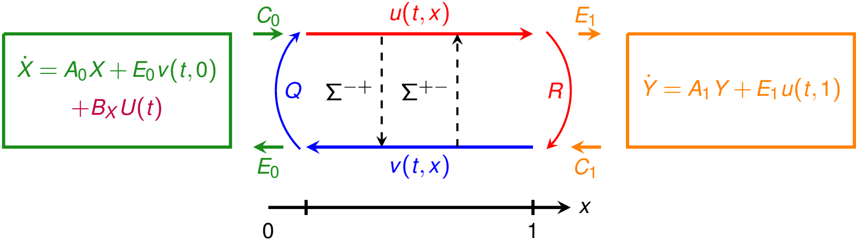
Strictly proper stabilizing control law!

Summary state-feedback

- Backstepping transformation to simplify the dynamics and the design of the control law.
- The regulation problem rewrites as a stabilization problem.
- Time-delay representation and frequency analysis.
- Low-pass filtering of the control law to make it **strictly proper**.

Observer design

$$
\begin{cases}
\dot{X}(t) = A_0 X(t) + E_0 v(t,0) + B_X U(t), \\
\partial_t u(t,x) + \Lambda^+ \partial_x u(t,x) = \Sigma^{++}(x) u(t,x) + \Sigma^{+-}(x) v(t,x), \\
\partial_t v(t,x) - \Lambda^- \partial_x v(t,x) = \Sigma^{-+}(x) u(t,x) + \Sigma^{--}(x) v(t,x), \\
u(t,0) = C_0 X(t) + Q v(t,0), \quad v(t,1) = R u(t,1) + C_1 Y(t), \\
\dot{Y}(t) = A_{11} Y(t) + E_1 u(t,1), \\
y = C_{mes} Y(t), \quad \dim(y) \geq \dim(u)
\end{cases}
$$



Problem statement

Design a **state observer** for the system based on the available measurement $y(t)$.

Methodology

- Backstepping transformation to simplify the dynamics and the design of the observer.

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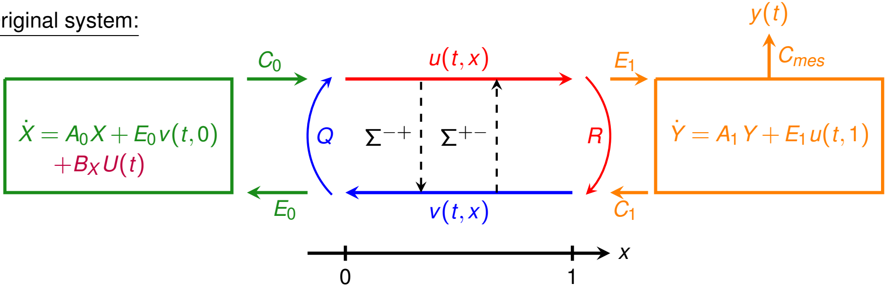
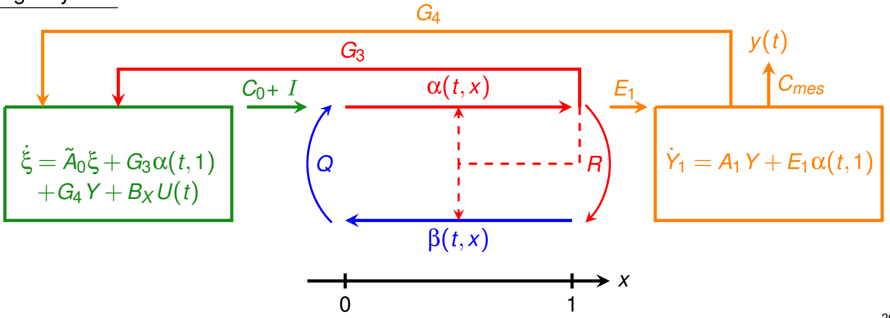
- Luenberger-like observer with operators $\mathcal{O}_i$ that need to be tuned.
- Design of the operators $\mathcal{O}_i$ to guarantee the exponential stability of the error system
- Convergence of the observer state to the real state.

Backstepping: Target system

Original system:

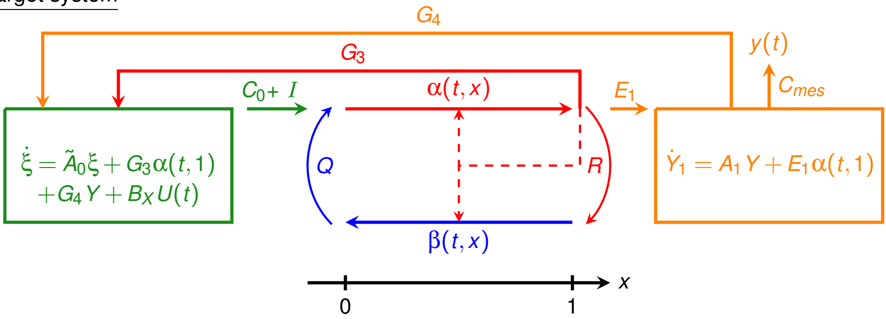


Target system



Backstepping: Target system

Target system



$$\dot{\xi}(t) = \tilde{A}_0\xi(t) + G_3\alpha(t,1) + G_4 Y(t) + B_X U(t),$$

$$\alpha(t,0) = Q\beta(t,0) + C_0\xi(t) + (Q\gamma_\beta(0) - \gamma_\alpha(0))Y(t) + \int_0^1 F^\alpha(y)\alpha(t,y) + F^\beta(y)\beta(t,y)dy,$$

$$\alpha_t(t,x) + \Lambda^+\alpha_x(t,x) = G_1(x)\alpha(t,1),$$

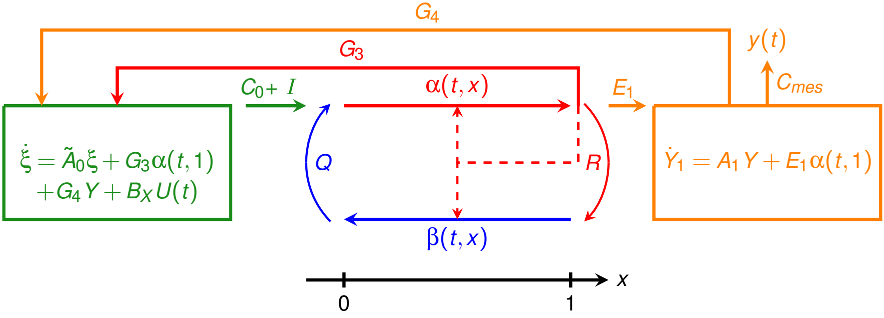
$$\beta_t(t,x) - \Lambda^-\beta_x(t,x) = G_2(x)\alpha(t,1),$$

$$\beta(t,1) = R\alpha(t,1), \ \dot{Y}(t) = A_1 Y(t) + E_1\alpha(t,1).$$

F^α **strictly lower triangular**

Backstepping: Target system

Target system



Advantages of the target system:

- Simplified in-domain couplings.
- Almost a "cascade structure" (except for the $\alpha(t,1)$-terms);
- Simplified observer design

Backstepping: Volterra transformation

$$
\begin{aligned}
X(t) &= \xi(t) - \int_0^1 L_1(y)\alpha(y) + L_2(y)\beta(y)dy, \\
u(t,x) &= \alpha(t,x) - \int_x^1 L^{\alpha\alpha}(x,y)\alpha(y)dy - \int_x^1 L^{\alpha\beta}(x,y)\beta(y)dy + \gamma_\alpha(x)Y(t), \\
v(t,x) &= \beta(t,x) - \int_x^1 L^{\beta\alpha}(x,y)\alpha(y)dy - \int_x^1 L^{\beta\beta}(x,y)\beta(y)dy + \gamma_\beta(x)Y(t), \\
Y(t) &= Y(t),
\end{aligned}
$$

- Triangular transformation: invertible.
- Kernels are bounded functions.

Observer equations

System (ξ, α, β, Y)

$$\dot{\xi}(t) = \tilde{A}_0\xi(t) + G_3\alpha(t,1) + G_4 Y(t) + B_X U(t),$$

$$\alpha(t,0) = Q\beta(t,0) + C_0\xi(t) + (Q\gamma_\beta(0) - \gamma_\alpha(0))Y(t) + \int_0^1 F^\alpha(y)\alpha(t,y) + F^\beta(y)\beta(t,y)dy,$$

$$\alpha_t(t,x) + \Lambda^+\alpha_x(t,x) = G_1(x)\alpha(t,1),$$

$$\beta_t(t,x) - \Lambda^-\beta_x(t,x) = G_2(x)\alpha(t,1),$$

$$\beta(t,1) = R\alpha(t,1), \; \dot{Y}(t) = A_1 Y(t) + E_1\alpha(t,1).$$

System $(\hat{\xi}, \hat{\alpha}, \hat{\beta}, \hat{Y})$: $\mathcal{O}_i$: **stable operators.**

$$\dot{\hat{\xi}}(t) = \tilde{A}_0\hat{\xi}(t) + G_3\hat{\alpha}(t,1) + G_4\hat{Y}(t) - \mathcal{O}_0(\tilde{y}),$$

$$\hat{\alpha}(t,0) = Q\hat{\beta}(t,0) + C_0\hat{\xi}(t) + (Q\gamma_\beta(0) - \gamma_\alpha(0))\hat{Y}(t) + \int_0^1 F^\alpha(y)\hat{\alpha}(t,y) + F^\beta(y)\hat{\beta}(t,y)dy - \mathcal{O}_1(\tilde{y}),$$

$$\hat{\alpha}_t(t,x) + \Lambda^+\hat{\alpha}_x(t,x) = G_1(x)\hat{\alpha}(t,1) - \mathcal{O}_\alpha(x,\tilde{y}),$$

$$\hat{\beta}_t(t,x) - \Lambda^-\hat{\beta}_x(t,x) = G_2(x)\hat{\alpha}(t,1) - \mathcal{O}_\beta(x,\tilde{y}),$$

$$\hat{\beta}(t,1) = R\hat{\alpha}(t,1), \quad \dot{\hat{Y}}(t) = A_1\hat{Y}(t) + E_1\hat{\alpha}(t,1) - L_1 C\tilde{y},$$

Error system

$$
\begin{aligned}
\dot{\tilde{\xi}}(t) &= \tilde{A}_0\tilde{\xi}(t) + G_3\tilde{\alpha}(t,1) + G_4\tilde{Y}(t) + B_X U(t)\mathcal{O}_0(\tilde{y}),\\
\tilde{\alpha}(t,0) &= C_0\tilde{\xi}(t) + Q\tilde{\beta}(t,0) + (Q\gamma_\beta(0) - \gamma_\alpha(0))\tilde{Y}(t)\\
&\quad + \int_0^1 F^\alpha(y)\tilde{\alpha}(t,y) + F^\beta(y)\tilde{\beta}(t,y)dy + \mathcal{O}_1(\tilde{y}),\\
\tilde{\alpha}_t(t,x) + \Lambda^+\tilde{\alpha}_x(t,x) &= G_1(x)\tilde{\alpha}(t,1) + \mathcal{O}_\alpha(x,\tilde{y})\\
\tilde{\beta}_t(t,x) - \Lambda^-\tilde{\beta}_x(t,x) &= G_2(x)\tilde{\alpha}(t,1) + \mathcal{O}_\beta(x,\tilde{y})\\
\tilde{\beta}(t,1) = R\tilde{\alpha}(t,1), &\quad \dot{\tilde{Y}}(t) = \tilde{A}_1\tilde{Y}(t) + E_1\tilde{\alpha}(t,1).
\end{aligned}
$$

- **Objective:** Tune the gains $\mathcal{O}_i$ such that the error system exponentially converges to zero.

Lemma: Cascade structure of the error system

If $\tilde{\xi}(t)$, $\tilde{\alpha}(t,1)$ and $\tilde{Y}(t)$ exponentially converge to zero, then the state $(\tilde{\xi},\tilde{\alpha},\tilde{\beta},\tilde{Y})$ exponentially converges to zero. This implies the convergence of the observer state to the real state.

Design of the operators O_i

- Laplace transform of $\dot{\tilde{Y}}(t) = \tilde{A}_1 \tilde{Y}(t) + E_1 \tilde{\alpha}(t,1)$:

$$(s\mathrm{Id} - \tilde{A}_1)\tilde{Y}(s) = E_1 \tilde{\alpha}(s,1) \rightarrow \boxed{\tilde{y}(s) = C_{mes}(s\mathrm{Id} - \tilde{A}_1)^{-1} E_1 \tilde{\alpha}(s,1)},$$

where $\tilde{A}_1$ is Hurwitz (**Assumption 4**) and $C_{mes}(s\mathrm{Id} - \tilde{A}_1)^{-1} E_1$ has no zeros in the RHP (**Assumption 2**)

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where $\tilde{A}_1$ is Hurwitz (**Assumption 4**) and $C_{mes}(s\mathrm{Id} - \tilde{A}_1)^{-1} E_1$ has no zeros in the RHP (**Assumption 2**)

- $P_1(s) = C_{mes}(s\mathrm{Id} - \tilde{A}_1)^{-1} E_1$ has a stable left-inverse (**Assumption 4**):

$$\boxed{\tilde{\alpha}(s,1) = P_1^-(s)\tilde{y}(s), \quad \tilde{Y}(s) = (s\mathrm{Id} - \tilde{A}_1)^{-1} E_1 P_1^-(s)\tilde{y}(s)}$$

Terms that are functions $\tilde{Y}$ and $\tilde{\alpha}(s,1)$ can be (exponentially) compensated using stable filters and values of $\tilde{y}(s)$.

Design of the operators $\mathcal{O}_i$

- Laplace transform of $\dot{\tilde{Y}}(t) = \tilde{A}_1 \tilde{Y}(t) + E_1 \tilde{\alpha}(t,1)$:

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Terms that are functions $\tilde{Y}$ and $\tilde{\alpha}(s,1)$ can be (exponentially) compensated using stable filters and values of $\tilde{y}(s)$.

- We have $\dot{\tilde{\xi}}(t) = \tilde{A}_0 \tilde{\xi}(t) + G_3 \tilde{\alpha}(t,1) + G_4 \tilde{Y}(t) + \mathcal{O}_0(\tilde{y})$

$$\mathcal{O}_0(\tilde{y}(s)) = -(G_3 P_1^-(s) + G_4(s\mathrm{Id} - \tilde{A}_1)^{-1} E_1 P_1^-(s))\tilde{y}(s) \Rightarrow (s\mathrm{Id} - \tilde{A}_0)\tilde{\xi}(s) = 0$$

Exponential convergence of $\tilde{\xi}$ to 0.

Design of the operators $\mathcal{O}_i$

$$\alpha(s,1) = P_1^-(s)\tilde{Y}(s), \quad \tilde{y}(s) = (s\mathrm{Id} - \tilde{A}_1)^{-1} E_1 P_1^-(s)\tilde{y}(s)$$

- We have $\tilde{\alpha}_t(t,x) + \Lambda^+ \tilde{\alpha}_x(t,x) = G_1(x)\tilde{\alpha}(t,1) + \mathcal{O}_\alpha(x,\tilde{y})$. Thus

$$\mathcal{O}_\alpha(x,\tilde{y}) = -G_1(x)P_1^-(s)\tilde{y}(s) \Rightarrow \tilde{\alpha}_t(t,x) + \Lambda^+ \tilde{\alpha}_x(t,x) = 0 \Rightarrow \boxed{\tilde{\alpha}_i(t,x) = \tilde{\alpha}_i(t - \frac{x}{\lambda_i}, 0)}.$$

Design of the operators $\mathcal{O}_i$

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- We have $\tilde{\beta}_t(t,x) - \Lambda^- \tilde{\beta}_x(t,x) = G_2(x)\tilde{\alpha}(t,1) + \mathcal{O}_\beta(x,\tilde{y})$. Thus

$$\mathcal{O}_\beta(x,\tilde{y}) = -G_2(x)P_1^-(s)\tilde{y}(s) \Rightarrow \tilde{\beta}_t(t,x) - \Lambda^-\tilde{\beta}_x(t,x) = 0$$

$$\Rightarrow \boxed{\beta_j(t,x) = \sum_{k=1}^{n} R_{jk}\tilde{\alpha}_k(t - \frac{1-x}{\mu_j}, 1)}.$$

An Integral Difference Equation

- The function $\tilde{\alpha}(t,0)$ verifies

$$\tilde{\alpha}_i(s,0) = ((Q\gamma_\beta(0) - \gamma_\alpha(0))\tilde{Y})_i + (\mathcal{O}_1(\tilde{y}))_i + \sum_{k=1}^{m}\sum_{\ell=1}^{n} Q_{ik}R_{k\ell}\mathrm{e}^{-\frac{s}{\mu_k}-\frac{s}{\lambda_\ell}}\tilde{\alpha}_\ell(s,0)$$
$$+ \int_0^1 \sum_{k=1}^{m}\sum_{\ell=1}^{n} F^{\beta}_{ik}(\nu)R_{k\ell}\mathrm{e}^{-\frac{s(1-\nu)}{\mu_k}}\tilde{\alpha}_\ell(s,1)d\nu$$
$$+ \int_0^1 \sum_{j=1}^{i} F^{\alpha}_{ij}(\nu)\sum_{k=1}^{m}\sum_{\ell=1}^{n} Q_{jk}R_{k\ell}\mathrm{e}^{-\frac{s\nu}{\lambda_j}}\mathrm{e}^{-\frac{s}{\mu_k}}\tilde{\alpha}_\ell(s,1)d\nu,$$

since F^{α} is strictly lower-triangular.

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$$+\int_0^1 \sum_{j=1}^{i} F_{ij}^{\alpha}(\nu)\sum_{k=1}^{m}\sum_{\ell=1}^{n} Q_{jk}R_{k\ell}\mathrm{e}^{-\frac{s\nu}{\lambda_j}}\mathrm{e}^{-\frac{s}{\mu_k}}\tilde{\alpha}_\ell(s,1)d\nu,$$

since F^α is strictly lower-triangular.

- Possible to recursively define $\mathcal{O}_1(\tilde{y})$ such that

$$\boxed{\tilde{\alpha}_i(t,0) = \sum_{k=1}^{m}\sum_{\ell=1}^{n} Q_{ik}R_{k\ell}\tilde{\alpha}_\ell(t - \frac{1}{\mu_k} - \frac{1}{\lambda_\ell}, 0)}$$

An Integral Difference Equation

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- Exponential stabilization of $\tilde{\alpha}(t,0)$ (and consequently of $\tilde{\alpha}(t,1)$) due to **Assumption 3**.

Convergence of the observer

- The states $\tilde{\alpha}(t,1)$ and $\tilde{\xi}$ exponentially converge to zero.

- We have $\dot{\tilde{Y}}(t) = \tilde{A}_1 \tilde{Y}(t) + E_1 \tilde{\alpha}(t,1)$ with $\tilde{A}_1$ Hurwitz. Thus the state $\tilde{Y}$ **exponentially converges to zero**.

- Stabilization of the error system.

Convergence of the observer

With the proposed operators $\mathcal{O}_0$, $\mathcal{O}_\alpha$, $\mathcal{O}_\beta$, $\mathcal{O}_1$, the observer state $(\hat{X}, \hat{u}, \hat{v}, \hat{Y}) = \mathcal{T}(\hat{\xi}, \hat{\alpha}, \hat{\beta}, \hat{Y})$ exponentially converges to (X, u, v, Y), $\mathcal{T}$ being the inverse backstepping transformation.

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- Possible to low-pass filter the measured output signal to use **strictly proper observer operators**

Convergence of the observer

- The states $\tilde{\alpha}(t,1)$ and $\tilde{\xi}$ exponentially converge to zero.
- We have $\dot{\tilde{Y}}(t) = \tilde{A}_1 \tilde{Y}(t) + E_1 \tilde{\alpha}(t,1)$ with $\tilde{A}_1$ Hurwitz. Thus the state $\tilde{Y}$ **exponentially converges to zero**.
- Stabilization of the error system.

Convergence of the observer

With the proposed operators $\mathcal{O}_0$, $\mathcal{O}_\alpha$, $\mathcal{O}_\beta$, $\mathcal{O}_1$, the observer state $(\hat{X}, \hat{u}, \hat{v}, \hat{Y}) = \mathcal{T}(\hat{\xi}, \hat{\alpha}, \hat{\beta}, \hat{Y})$ exponentially converges to (X, u, v, Y), $\mathcal{T}$ being the inverse backstepping transformation.

- Possible to low-pass filter the measured output signal to use **strictly proper observer operators**
- The proposed observer could be combined with the previous state-feedback laws to obtain a strictly proper **output-feedback controller**.

Simulation results

Parameters:

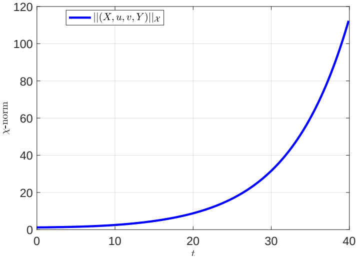
$\lambda = 2$, $\mu = 0.7$, $\sigma^{+-} = 1$, $\sigma^{-+} = 0.5$, $\rho = 0.5$, $q = 1.2$.

ODE dynamics in dimension $n = 4, m = 3, c = 2$

$$A_0 = \begin{bmatrix} 0 & 0.14 & 0 & 0.1 \\ 0 & 0 & 0.14 & 0 \\ 0.29 & -0.43 & 0.57 & 0.2 \\ 0 & 0 & 0 & -1.1 \end{bmatrix}, B_0 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -0.5 \end{bmatrix}^T, E_0 = \begin{bmatrix} 2 \\ -1 \\ 0.1 \\ 0 \end{bmatrix}, C_{11} = \begin{bmatrix} 0 \\ 1 \\ 0.5 \end{bmatrix}^T$$

$$A_{11} = \begin{bmatrix} 0.29 & 0.14 & 0 \\ 0.14 & 0 & 0.1 \\ 0 & 0 & -0.9 \end{bmatrix}, E_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$



Unstable system in open-loop.

We want to reject a sinusoidal disturbance

Simulation results

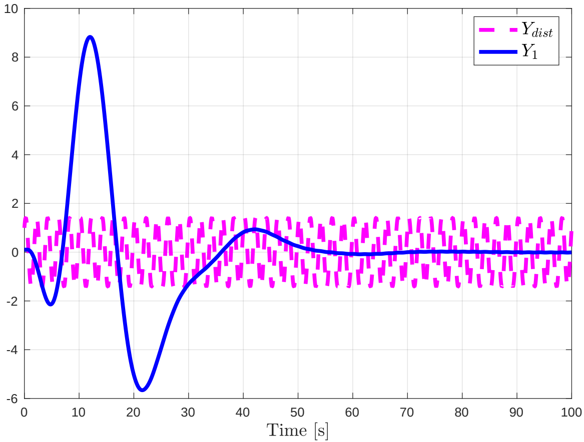


Figure: Evolution of the distal ODE state $Y_1(t)$ (blue) in the presence of a disturbance Y_{dist}

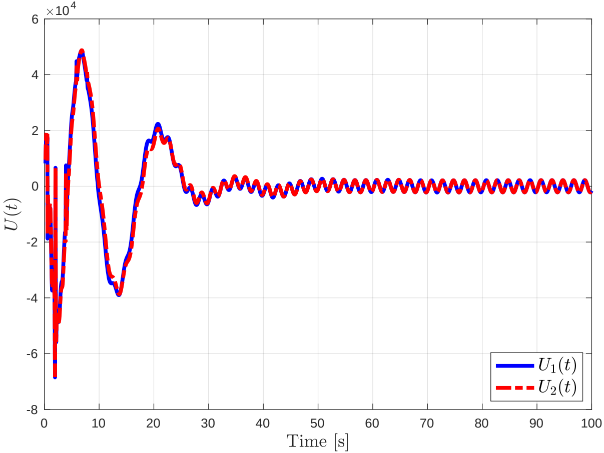


Figure: Evolution of the control inputs $U_1(t)$ (blue) and $U_2(t)$ (red)

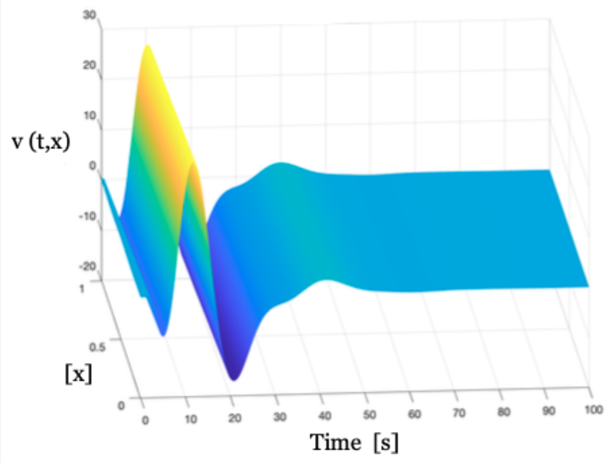


Figure: Evolution of the PDE state $v(t,x)$

Simulation results

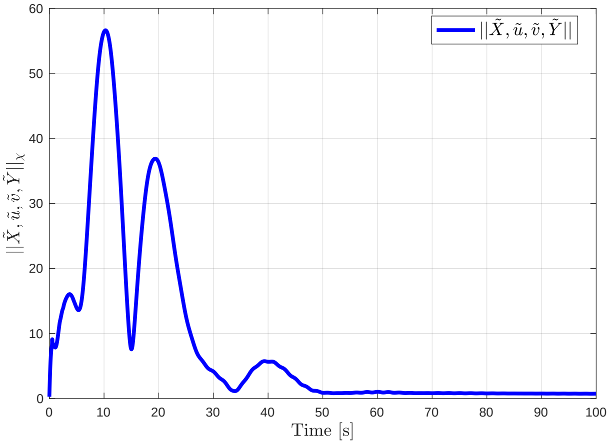


Figure: Evolution of the norm of the error state

Conclusions and perspectives

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  - Backstepping transformation to simplify the structure of the system
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  - Leverage the different assumptions?
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