Output Regulation for a class of linear ODE-Hyperbolic PDE-ODE systems

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Motivation

*Why hyperbolic systems?*

- **Conservation/balance** of scalar quantities when taking into account:
  - Evolution (e.g., *transport*) of conserved quantities in space and time
  - Finite *speed of propagation* (vs. heat equation)
Motivation

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  - Evolution (e.g., transport) of conserved quantities in space and time
  - Finite **speed of propagation** (vs. heat equation)

- **Natural representation** for some industrial processes for which you have
  - long distances (e.g. pipeline)
  - slow propagation speeds (e.g. traffic)
  - spatially dependent characteristics (e.g. composite materials)
  - anisotropic behavior (e.g. ferromagnetism)

Mathematically, this may look something like:

\[
\frac{\partial}{\partial t} \rho(t,x) = \nabla f(t,x) + S(t,x), \quad \forall (t,x) \in [0,T] \times \Omega,
\]

where \( \rho \) is the conserved quantity, \( f \) is a flux density and \( S \) is a source term.
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- **Multiple problems:** stabilization, control, observability, parameter estimation...
  - Wave equation: \( \partial_{tt} w(t, x) - c^2 \partial_{xx} w(t, x) = 0 \).
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Mathematically, this may look something like:

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\partial_t \rho(t, x) = \nabla f(t, x) + S(t, x), \quad \forall (t, x) \in [0, T] \times \Omega,
\]

where \( \rho \) is the **quantity conserved**, \( f \) is a **flux density** and \( S \) is a **source term**.

Motivation

Many physical laws are conservation/balance laws, e.g. mass, charge, energy, momentum [Bastin, Coron; 2016]
**Networks of hyperbolic systems**

*Why coupled and interconnected hyperbolic systems?*

- Conservation/balance laws rarely appear isolated
  - Navier-Stokes $\rightarrow$ mass + energy + momentum
  - Propagation phenomena rarely occur in a single direction

- Systems modeled by hyperbolic PDEs do not exist in isolation, e.g.:
  - Electric transmission networks $\rightarrow$ interconnection of individual transmission lines
  - Mechanical vibrations in drilling devices $\rightarrow$ interconnection of different pipes

- Possible coupling with ODEs
  - actuator dynamics (e.g. pump, converter)
  - load dynamics (e.g. valve, motor)
  - sensor dynamics (e.g. flow-rate sensor, tachometer)
Examples of interconnected ODE-PDEs-ODE systems

Applications: drilling systems, deepwater construction vessels [Wang et al.]
Interconnections of hyperbolic PDEs and ODEs are not a new problem.

Many **constructive** control results based on the **backstepping approach**, e.g.:

- Seminal paper [Krstic and Smyshlyaev, 2008]: re-interpretation of the classical Finite Spectrum Assignment [Manitius and Olbrot, 1979] (ODE + input delays)
- Time-varying delays [Bekiaris-Liberis and Krstic, 2013, Bresch-Pietri, 2012],
- Cascades of PDEs [Auriol et al., 2019]
Interconnections of hyperbolic PDEs and ODEs are not a new problem.

- Many constructive control results based on the backstepping approach, e.g.:
  - Seminal paper [Krstic and Smyshlyaev, 2008]: re-interpretation of the classical Finite Spectrum Assignment [Manitius and Olbrot, 1979] (ODE + input delays)
  - Time-varying delays [Bekiaris-Liberis and Krstic, 2013, Bresch-Pietri, 2012],
  - Cascades of PDEs [Auriol et al., 2019]

- For fully-interconnected (non-cascaded) systems some examples include:
  - stabilizing state-feedback control law in [Di Meglio et al., 2018, Wang et al., 2018]
  - output regulation for coupled linear wave–ODE systems [Deutscher and Gabriel, 2021]
Interconnected PDE-ODE systems: control design

- For ODE-hyperbolic PDE-ODE systems with full interconnections (non-cascade):
  - state feedback in [Bou Saba et al., 2017] for scalar PDE system (invertible input matrix)
  - output-feedback controller based on a Byrnes-Isidori normal form for the proximal ODE, as well as a relative degree one condition in [Deutscher et al., 2018]
  - strictly-proper state-feedback control law for scalar PDE in [Bou Saba et al., 2019] requiring minimum-phase assumption (not relative degree 1)
  - extended to output-feedback control for scalar PDE in [Wang and Krstic, 2020]
  - stabilizing observer-controller robust to delays in the case of a scalar proximal ODE in [Di Meglio et al., 2020]

- Some recent results have also been obtained for interconnected PDE systems with non-linear ODEs [Irscheid et al., 2021]
What you will see in this presentation

- **Output regulation** of a general class of ODE-PDE-ODE system
  - Finite-dimensional exo-system representing the reference trajectory and disturbance dynamics.
  - Backstepping approach: integral change of coordinates
  - Time delay representation and frequency analysis
  - Stabilizing control law in the absence of the disturbance
Content of the presentation

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  - Backstepping approach: integral change of coordinates
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- **A robustification procedure**
  - Low-pass filter to make the control law strictly proper
  - Frequency analysis
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  - Stabilizing control law in the absence of the disturbance

- A **robustification procedure**
  - Low-pass filter to make the control law strictly proper
  - Frequency analysis

- **Observer design**
  - Backstepping approach to simplify the dynamics
  - Luenberger-like observer with tuning operators
  - Frequency analysis
  - Output-feedback control law
System under consideration: ODE-PDE-ODE

\[
\dot{X}(t) = A_0 X(t) + E_0 v(t, 0) + B_X U(t),
\]
\[
\partial_t u(t, x) + \Lambda^+ \partial_x u(t, x) = \Sigma^{++}(x) u(t, x) + \Sigma^{+-}(x) v(t, x),
\]
\[
\partial_t v(t, x) - \Lambda^- \partial_x v(t, x) = \Sigma^{-+}(x) u(t, x) + \Sigma^{--}(x) v(t, x),
\]
\[
u(t, 0) = C_0 X(t) + Q v(t, 0), \quad v(t, 1) = R u(t, 1) + C_1 Y(t),
\]
\[
\dot{Y}(t) = A_{11} Y(t) + E_1 u(t, 1),
\]

Measurement: \( y(t) = C_{\text{mes}} Y(t) \)

Same concepts for scalar and non-scalar PDEs systems
System under consideration: ODE-PDE-ODE

\[
\begin{aligned}
\dot{X}(t) &= A_0 X(t) + E_0 v(t, 0) + B_X U(t), \\
\partial_t u(t, x) + \lambda \partial_x u(t, x) &= \sigma^{++}(x) u(t, x) + \sigma^{+-}(x) u(t, x), \\
\partial_t v(t, x) - \mu \partial_x v(t, x) &= \sigma^{-+}(x) u(t, x) + \sigma^{--}(x) v(t, x), \\
u(t, 0) &= C_0 X(t) + q v(t, 0), \\
v(t, 1) &= \rho u(t, 1) + C_1 Y(t), \\
\dot{Y}(t) &= A_1 Y(t) + E_1 u(t, 1),
\end{aligned}
\]

- **Measurement:** \( y(t) = C_{\text{mes}} Y(t) \)
- Same concepts for scalar and non-scalar PDEs systems
- Diagonal terms can be removed with exp. change of coordinates
System under consideration: ODE-PDE-ODE

\[\begin{align*}
\dot{X}(t) &= A_0 X(t) + E_0 \nu(t,0) + B_X U(t), \\
\partial_t u(t,x) + \lambda \partial_x u(t,x) &= \sigma^{++}(x) u(t,x) + \sigma^{+-}(x) u(t,x), \\
\partial_t \nu(t,x) - \mu \partial_x \nu(t,x) &= \sigma^{-+}(x) u(t,x) + \sigma^{--}(x) \nu(t,x), \\
u(t,0) &= C_0 X(t) + q \nu(t,0), \quad \nu(t,1) = \rho \nu(t,1) + C_1 Y(t), \\
\dot{Y}(t) &= A_1 Y(t) + E_1 u(t,1),
\end{align*}\]

- **Measurement:** \( y(t) = C_{\text{mes}} Y(t) \)
- Same concepts for scalar and non-scalar PDEs systems
- Diagonal terms can be removed with exp. change of coordinates
- Initial conditions in \( H^1 \) with appropriate compatibility conditions \( \rightarrow \) well-posedness
System under consideration: ODE-PDE-ODE

\[
\begin{align*}
\dot{X}(t) &= A_0 X(t) + E_0 v(t,0) + B_X U(t), \\
\partial_t u(t,x) + \lambda \partial_x u(t,x) &= \sigma^{++}(x) u(t,x) + \sigma^{+-}(x) u(t,x), \\
\partial_t v(t,x) - \mu \partial_x v(t,x) &= \sigma^{-+}(x) u(t,x) + \sigma^{--}(x) v(t,x), \\
u(t,0) &= C_0 X(t) + q v(t,0), \quad v(t,1) = \rho u(t,1) + C_1 Y(t), \\
\dot{Y}(t) &= A_1 Y(t) + E_1 u(t,1),
\end{align*}
\]

- **Measurement:** \( y(t) = C_{\text{mes}} Y(t) \)
- Same concepts for scalar and non-scalar PDEs systems
- Diagonal terms can be removed with exp. change of coordinates
- Initial conditions in \( H^1 \) with appropriate compatibility conditions \( \rightarrow \text{well-posedness} \)
- Stabilization in the sense of the \( L^2 \)-norm
System under consideration: well-posedness and stabilization objective

\[
\begin{align*}
\dot{X}(t) &= A_0 X(t) + E_0 v(t,0) + B_X U(t), \\
\partial_t u(t,x) + \lambda \partial_x u(t,x) &= \sigma^-(x) u(t,x), \\
\partial_t v(t,x) - \mu \partial_x v(t,x) &= \sigma^+(x) u(t,x), \\
\dot{u}(t,0) &= C_0 X(t) + q v(t,0), \quad \dot{v}(t,1) = \rho u(t,1) + C_1 Y(t), \\
\dot{Y}(t) &= A_1 Y(t) + E_1 u(t,1),
\end{align*}
\]

Well-posedness in open-loop

For every initial condition \((X_0, u_0, v_0, Y_0) \in \mathbb{R}^p \times H^1([0,1], \mathbb{R}^2) \times \mathbb{R}^q\) that verifies the compatibility conditions

\[
\begin{align*}
u_0(0) &= C_0 X(t) + Qv_0(0), \quad v_0(1) = Ru_0(1) + C_1 Y(t),
\end{align*}
\]

there exists one and one only \((X, u, v, Y)\) which is a solution to the open-loop Cauchy problem (i.e., \(U \equiv 0\)).

Moreover, there exists \(\kappa_0 > 0\) such that for every \((X_0, u_0, v_0, Y_0) \in \mathbb{R}^p \times H^1([0,1], \mathbb{R}^2) \times \mathbb{R}^q\) satisfying the compatibility conditions, the unique solution verifies

\[
\|(X(t), u(t,\cdot), v(t,\cdot), Y(t))\|_\chi \leq \kappa_0 e^{\kappa_0 t} \|(X_0, u_0, v_0, Y_0)\|_\chi, \quad \forall t \in [0,\infty).
\]

where \(\|(X(t), u(t,\cdot), v(t,\cdot), Y(t))\|_\chi = \sqrt{\|X(t)\|^2_{\mathbb{R}^p} + \|u(t,\cdot)\|^2_{L^2} + \|v(t,\cdot)\|^2_{L^2} + \|Y(t)\|^2_{\mathbb{R}^q}}\).
System under consideration: well-posedness and stabilization objective

\[\begin{align*}
\dot{X}(t) &= A_0 X(t) + E_0 v(t, 0) + B_X U(t), \\
\partial_t u(t, x) + \lambda \partial_x u(t, x) &= \sigma^-(x) u(t, x), \\
\partial_t v(t, x) - \mu \partial_x v(t, x) &= \sigma^+(x) u(t, x), \\
u(t, 0) &= C_0 X(t) + q v(t, 0), \\
v(t, 1) &= \rho u(t, 1) + C_1 Y(t), \\
\dot{Y}(t) &= A_1 Y(t) + E_1 u(t, 1),
\end{align*}\]

Stabilization objective

Design a continuous control input that exponentially stabilizes the system in the sense of the \(L^2\)-norm, i.e. there exist \(\kappa_0\) and \(\nu > 0\) such that for any initial condition \((X_0, u_0, v_0, Y_0) \in \mathbb{R}^p \times H^1([0, 1], \mathbb{R}^2) \times \mathbb{R}^q\), we have

\[\|\big(X(t), u(t, \cdot), v(t, \cdot), Y(t)\big)\|_\chi \leq \kappa_0 e^{-\nu t} \|\big(X_0, u_0, v_0, Y_0\big)\|_\chi, \quad 0 \leq t\]
Output-regulation problem

\[ \dot{X} = A_0 X + E_0 v(t, 0) + B_x U(t) \]

Augmented variable: \( Y(t) = (Y_1^T(t), Y_2^T(t))^T \)
- \( Y_1 \) is the "real" ODE state
- \( Y_2 \) is an exogenous input: disturbance \( Y_{\text{dist}} \) and/or a reference trajectory \( Y_{\text{ref}} \)

\[ \dot{Y}(t) = A_1 Y(t) + \begin{pmatrix} E_1 \\ 0_{q_2 \times 1} \end{pmatrix} u(t, 1), \text{ with } A_1 = \begin{pmatrix} A_{11} & A_{12} \\ 0_{q_2 \times q_1} & A_{22} \end{pmatrix}, \]
Output-regulation problem

\[ \dot{X} = A_0 X + E_0 \nu(t,0) + B_X U(t) \]

Augmented variable: \[ Y(t) = (Y_1^T(t), Y_2^T(t))^\top \]
- \( Y_1 \) is the "real" ODE state
- \( Y_2 \) is an exogenous input: disturbance \( Y_{\text{dist}} \) and/or a reference trajectory \( Y_{\text{ref}} \)

\[ \dot{Y}(t) = A_1 Y(t) + \begin{pmatrix} E_1 \\ 0_{q_2 \times 1} \end{pmatrix} u(t,1) \]

Virtual output: \[ \varepsilon(t) = C_e Y(t) = \begin{pmatrix} C_{e1} & C_{e2} \end{pmatrix} Y(t) \]

Control objective

Design a control law \( U(t) \) s.t. the virtual output \( \varepsilon(t) \) exp. converges to zero.
Output-regulation problem

\[ \dot{X} = A_0 X + E_0 v(t, 0) + B_1 U(t) \]

Augmented variable: \( Y(t) = \begin{pmatrix} Y_1^\top(t) & Y_2^\top(t) \end{pmatrix}^\top \)

- \( Y_1 \) is the "real" ODE state
- \( Y_2 \) is an **exogenous input**: disturbance \( Y_{\text{dist}} \) and/or a **reference trajectory** \( Y_{\text{ref}} \)

\[
\dot{Y}(t) = A_1 Y(t) + \begin{pmatrix} E_1 \\ 0_{q_2 \times 1} \end{pmatrix} u(t, 1), \quad \text{with } A_1 = \begin{pmatrix} A_{11} & A_{12} \\ 0_{q_2 \times q_1} & A_{22} \end{pmatrix},
\]

Virtual output: \( \varepsilon(t) = C_e Y(t) = \begin{pmatrix} C_{e1} & C_{e2} \end{pmatrix} Y(t) \)

**Output regulation problem:** \( C_{e1} \not\equiv 0 \), and \( C_{e2} \equiv 0 \): we want to regulate to zero a linear combination of components of \( Y_1(t) \) in the presence of a disturbance \( Y_2(t) \).
Output-regulation problem

\[
\dot{X} = A_0 X + E_0 v(t, 0) + B_X U(t)
\]

Augmented variable:

\[
Y(t) = \left( Y_1^T(t), Y_2^T(t) \right)^T
\]

- \( Y_1 \) is the "real" ODE state
- \( Y_2 \) is an exogenous input: disturbance \( Y_{\text{dist}} \) and/or a reference trajectory \( Y_{\text{ref}} \)

Virtual output:

\[
\varepsilon(t) = C_e Y(t) = (C_{e1} \quad C_{e2}) Y(t)
\]

Output tracking problem: \( C_{e1,i} - C_{e2,j} = 0 \) (other components = 0): we want the \( i^{\text{th}} \) component of the output \( Y_1 \) to converge towards the \( j^{\text{th}} \) component of a known trajectory \( Y_2 \).
Structural assumptions

\[ \dot{X} = A_0 X + E_0 v(t,0) + B X U(t) \]

Assumption 1: Stabilizability

The pairs \((A_0, B_0)\) and \((A_{11}, E_1)\) are **stabilizable**, i.e. there exist \(F_0 \in \mathbb{R}^{r \times p}\), \(F_1 \in \mathbb{R}^{n \times q_1}\) such that \(\tilde{A}_0 = A_0 + B X F_0\) and \(\tilde{A}_{11} = A_{11} + E_1 F_1\) are Hurwitz.

- Classical requirement found in most of the papers dealing with ODE-PDE-ODE
- Not overly conservative (necessary to stabilize \(Y\), slightly conservative for \(X\)).
Structural assumptions

\[ \dot{X} = A_0 X + E_0 v(t, 0) + B_X U(t) \]

**Assumption 2**
For all \( s \in \mathbb{C}_0 \), the matrices \((A_0, B_X, C_0)\) satisfy

\[
\text{rank} \begin{pmatrix} s \text{id} - A_0 & B_X \\ C_0 & 0_{n\times r} \end{pmatrix} = p + 1 = p + n.
\]

- The function \( P_0(s) = C_0(s \text{id} - \bar{A}_0)^{-1} B_X \) does not have any zeros in \( \mathbb{C}^+ \)
- **Stable right inverse** of \( P_0(s) \)
Structural assumptions

\[
\dot{X} = A_0 X + E_0 v(t, 0) + B_X U(t)
\]

Assumption 3: Delay-robustness

The coefficients \( \rho \) and \( q \) verify \( |\rho q| < 1 \).

- No asymptotic chain of eigenvalues with non-negative real parts
- Necessary for (delay-) robust stabilization
Structural assumptions

\[ \dot{X} = A_0 X + E_0 v(t, 0) + B_X U(t) \]

Assumption 4: detectability

The pairs \((A_1, C), (A_0, C_0)\) are detectable (i.e. there exist \(L_0 \in \mathbb{R}^{p \times n}\) and \(L_1 \in \mathbb{R}^{q \times d}\) such that \(\tilde{A}_1 \doteq A_1 + L_1 C_{mes}\) and \(\tilde{A}_0 \doteq A_0 + L_0 C_0\) are Hurwitz).

- Classical requirement found in most of the papers dealing with ODE-PDE-ODE
- Not overly conservative (necessary for reconstruction of \(X_0\), slightly conservative for \(Y\)).
Structural assumptions

\[ \dot{X} = A_0 X + E_0 v(t, 0) + B_X U(t) \]

\[ \dot{Y} = A_1 Y + \begin{pmatrix} E_1 \\ 0 \end{pmatrix} u(t, 1) \]

**Assumption 5**

For all \( s \in \mathbb{C}^+ \), the matrices \( (A_1, E_1, C) \) satisfy

\[ \text{rank} \left( \begin{pmatrix} s \text{Id} - A_1 & E_1 \\ C_{mes} & 0 \end{pmatrix} \right) = q + 1 = q + n. \]

- Necessary to independently reconstruct the different PDE boundary values by inverting the \( Y \) dynamics.
- The function \( P_1(s) \doteq C_{mes}(s \text{Id} - \tilde{A}_1)^{-1} E_1 \) does not have any zeros in \( \mathbb{C}^+ \)
- **Stable left-inverse** of \( P_1(s) \)
Structural assumptions

\[
\dot{X} = A_0 X + E_0 v(t, 0) + B_X U(t)
\]

Assumption 6

The matrix $A_{22}$ is marginally stable, i.e., all its eigenvalues have zero real parts. There exist matrices $T_a \in \mathbb{R}^{q_1 \times q_2}, F_a \in \mathbb{R}^{n \times q_2}$ solutions to the regulator equations:

\[
\begin{aligned}
-A_{11} T_a + T_a A_{22} + A_{12} &= -E_1 F_a, \\
-C_{e1} T_a + C_{e2} &= 0.
\end{aligned}
\]

- Non-resonance condition.
- $A_{11}$ and $A_{22}$ have disjoint spectra, and the number of outputs we regulate is coherent with the number of inputs.
- The matrices $T_a, F_a$ can be computed using a Schur triangulation.
Control design: strategy.

- Backstepping transformation to simplify the dynamics and the design of the control law.

- The regulation problem rewrites as a stabilization problem.

- Time-delay representation and frequency analysis.

- Low-pass filtering of the control law to make it strictly proper.
Backstepping methodology

- Map the original system to a target system for which the stability analysis is easier.
- Variable change: integral transformation, classically Volterra transform of the second kind

\[
\alpha(t, x) = u(t, x) - \int_0^x K^{uu}(x, \xi) u(t, \xi) + K^{uv}(x, \xi) v(t, \xi) d\xi,
\]

\[
\beta(t, x) = v(t, x) - \int_0^x K^{vu}(x, \xi) u(t, \xi) + K^{vv}(x, \xi) v(t, \xi) d\xi,
\]

Condensed form: \[
\gamma(t, x) = w(t, x) - \int_0^x K(x, y) w(t, y) dy.\]

Limitations

- Choice of an adequate target system.
- Proof of existence and invertibility of an adequate backstepping transform.
Objective: Move the in-domain coupling terms at the actuated boundary.

\[ u_t(t, x) + \lambda u_x(t, x) = \sigma^+ v(t, x), \]

\[ v_t(t, x) - \mu v_x(t, x) = \sigma^- u(t, x). \]

\[ u(t, 0) = qv(t, 0) + U(t) \]

\[ v(t, 1) = \rho u(t, 1) \]
Objective: Move the in-domain coupling terms at the actuated boundary.

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\[ u(t, 0) = qv(t, 0) + U(t) \]
\[ v(t, 1) = \rho u(t, 1) \]

\[ \alpha_t(t, x) + \lambda \alpha_x(t, x) = 0, \]
\[ \beta_t(t, x) - \mu \beta_x(t, x) = 0. \]
**Objective:** Move the in-domain coupling terms at the actuated boundary.

\[
\begin{align*}
    u_t(t, x) + \lambda u_x(t, x) &= \sigma^+ v(t, x), \\
    v_t(t, x) - \mu v_x(t, x) &= \sigma^- u(t, x).
\end{align*}
\]

\[
\begin{align*}
    \alpha_t(t, x) + \lambda \alpha_x(t, x) &= 0, \\
    \beta_t(t, x) - \mu \beta_x(t, x) &= 0.
\end{align*}
\]

\[
\begin{align*}
    u(t, 0) &= qv(t, 0) + U(t), \\
    v(t, 1) &= \rho u(t, 1).
\end{align*}
\]

\[
\begin{align*}
    \alpha(t, 0) &= q\beta(t, 0) + U(t) \\
    - \int_0^1 N^\alpha(\xi) \alpha(t, \xi) + N^\beta(\xi) \beta(t, \xi) \, d\xi.
\end{align*}
\]

\[
\begin{align*}
    \beta(t, 1) &= \rho \alpha(t, 1)
\end{align*}
\]
Objective: Move the in-domain coupling terms at the actuated boundary.

\[
\begin{align*}
    u_t(t, x) + \lambda u_x(t, x) &= \sigma^+ v(t, x), \\
    v_t(t, x) - \mu v_x(t, x) &= \sigma^- u(t, x).
\end{align*}
\]

\[
\begin{align*}
    u(t, 0) &= qv(t, 0) + U(t) \\
    v(t, 1) &= \rho u(t, 1)
\end{align*}
\]

\[
\begin{align*}
    \alpha_t(t, x) + \lambda \alpha_x(t, x) &= 0, \\
    \beta_t(t, x) - \mu \beta_x(t, x) &= 0.
\end{align*}
\]

Natural control law

\[
U(t) = -q\beta(t, 0) + \int_0^1 \left( N^\alpha(\xi)\alpha(t, \xi) + N^\beta(\xi)\beta(t, \xi) \right) d\xi.
\]
Backstepping: Volterra transformation

\[ X(t) = \xi(t) + \int_{0}^{1} M^{12}(y)\alpha(t, y) + M^{13}(y)\beta(t, y) dy + \begin{bmatrix} M^{14} & M^{15} \end{bmatrix} \eta(t), \]

\[ u(t, x) = \alpha(t, x) + \int_{x}^{1} M^{22}(x, y)\alpha(y) + M^{23}(x, y)\beta(y) dy + \begin{bmatrix} M^{24}(x) & M^{25}(x) \end{bmatrix} \eta(t), \]

\[ v(t, x) = \beta(t, x) + \int_{x}^{1} M^{32}(x, y)\alpha(y) + M^{33}(x, y)\beta(y) dy + \begin{bmatrix} M^{34}(x) & M^{35}(x) \end{bmatrix} \eta(t), \]

\[ Y(t) = \eta(t). \]

- Triangular transformation: invertible.

\[
\begin{pmatrix}
X(t) \\
u(t, x) \\
v(t, x) \\
Y(t)
\end{pmatrix} = \begin{pmatrix}
\text{Id} & \int_{0}^{1} M^{12}(y) dy & \int_{0}^{1} M^{13}(y) dy & \begin{bmatrix} M^{14} & M^{15} \end{bmatrix} \\
0 & \text{Id} + \int_{x}^{1} M^{22}(x, y) dy & \int_{x}^{1} M^{23}(x, y) dy & \begin{bmatrix} M^{24}(x) & M^{25}(x) \end{bmatrix} \\
0 & \int_{x}^{1} M^{32}(x, y) dy & \text{Id} + \int_{x}^{1} M^{33}(x, y) dy & \begin{bmatrix} M^{34}(x) & M^{35}(x) \end{bmatrix} \\
0 & 0 & 0 & \text{Id}
\end{pmatrix} \begin{pmatrix}
\xi(t) \\
\alpha(t, x) \\
\beta(t, x) \\
\eta(t)
\end{pmatrix}
\]

- Kernels are bounded functions.

- Unique solution due to the rank condition on \( C_{0} \).
Backstepping: Target system

Original system:

\[
\dot{X} = A_0 X + E_0 v(t, 0) + B_X U(t)
\]

Target system:

\[
\dot{\xi} = A_0 \xi + E_0 \beta(t, 0) + O(\xi, \alpha, \beta, \eta) + B_X U(t)
\]

\[
\dot{\eta} = \bar{A}_1 \eta + \begin{pmatrix} E_1 \\ 0 \end{pmatrix} \alpha(t, 1)
\]
Backstepping: Target system

Original system:

\[
\begin{align*}
\dot{X}(t) &= A_0 X(t) + E_0 \nu(t,0) + B_X U(t), \\
\partial_t u(t,x) + \Lambda^+ \partial_x u(t,x) &= \sigma^+ (x) u(t,x), \\
\partial_t \nu(t,x) - \Lambda^- \partial_x \nu(t,x) &= \sigma^- (x) u(t,x), \\
u(t,0) &= C_0 X(t) + q \nu(t,0), \quad \nu(t,1) = \rho \nu(t,1) + C_1 Y(t), \\
\dot{Y}(t) &= A_1 Y(t) + (E_1 \ 0)^\top u(t,1), \\
\end{align*}
\]

Target system:

\[
\begin{align*}
\dot{\xi}(t) &= \bar{A}_0 \xi(t) + \bar{E}_1 \alpha(t,1) + \bar{E}_0 \beta(t,0) + M \eta(t) \\
&\quad + \int_0^1 M \alpha(y) \alpha(t,y) + M \beta(y) \beta(t,y) dy + B_X \bar{U}(t), \\
\partial_t \alpha(t,x) + \Lambda^+ \partial_x \alpha(t,x) &= 0, \\
\partial_t \beta(t,x) - \Lambda^- \partial_x \beta(t,x) &= 0, \\
\alpha(t,0) &= C_0 \xi(t) + q \beta(t,0), \quad \beta(t,1) = \rho \alpha(t,1), \\
\dot{\eta}(t) &= \bar{A}_1 \eta(t) + (E_1 \ 0)^\top \alpha(t,1), \\
\end{align*}
\]

\[
\bar{A}_0 = A_0 + B_X F_0, \quad \bar{A}_1 = \begin{pmatrix} A_{11} + E_1 F_1 & A_{12} + E_1 (F_a + F_1 T_a) \\ 0 & A_{22} \end{pmatrix}
\]

Advantages of the target system:

- Simplified in-domain couplings.
- Almost a "cascade structure"
- To stabilize the whole system, we can focus on the stabilization of \( \xi \).
A cascade structure

\[
\begin{align*}
\dot{\xi}(t) &= \tilde{A}_0 \xi(t) + \tilde{E}_1 \alpha(t, 1) + \tilde{E}_0 \beta(t, 0) + M\eta(t) \\
&\quad + \int_0^1 M\alpha(y) \alpha(t, y) + M\beta(y) \beta(t, y) dy + B_x \tilde{U}(t), \\
\partial_t \alpha(t, x) + \Lambda^+ \partial_x \alpha(t, x) &= 0, \\
\partial_t \beta(t, x) - \Lambda^- \partial_x \beta(t, x) &= 0, \\
\alpha(t, 0) &= C_0 \xi(t) + q\beta(t, 0), \quad \beta(t, 1) = \rho \alpha(t, 1), \\
\dot{\eta}(t) &= \tilde{A}_1 \eta(t) + (E_1 \quad 0)^\top \alpha(t, 1),
\end{align*}
\]

Stability and regulation

If \( C_0 \xi \) exp. converges to zero, then \( \varepsilon(t) \to 0 \). Furthermore, the trajectories are bounded.
Assumption 6

The matrix $A_{22}$ is marginally stable, i.e., all its eigenvalues have zero real parts. There exist matrices $T_a \in \mathbb{R}^{q_1 \times q_2}$, $F_a \in \mathbb{R}^{n \times q_2}$ solutions to the regulator equations:

\[
\begin{align*}
- A_{11} T_a + T_a A_{22} + A_{12} &= - E_1 F_a, \\
- C e_1 T_a + C e_2 &= 0.
\end{align*}
\]

Stability and regulation

If $\xi$ exp. converges to zero, then $\varepsilon(t) \rightarrow 0$. Furthermore, the trajectories are bounded.

Proof: If $C_0 \xi$ converges to zero, then so does $\|(\alpha, \beta)\|_{L^2}$.

We have

\[
\begin{align*}
\dot{Y}_1 &= (A_{11} + E_1 F_1) Y_1(t) + (A_{12} + E_1 (F_a + F_1 T_a)) Y_2(t) + E_1 \alpha(t, 1) \\
&= (A_{11} + E_1 F_1) Y_1(t) + (A_{11} T_a - E_1 F_a - T_a A_{22}) Y_2(t) + E_1 (F_a + F_1 T_a) Y_2(t) + E_1 \alpha(t, 1), \\
\Rightarrow (Y_1 + T_a Y_2)(t) &= \tilde{A}_{11} (Y_1 + T_a Y_2) + E_1 \alpha(t, 1).
\end{align*}
\]
A cascade structure

Assumption 6

The matrix $A_{22}$ is marginally stable, i.e., all its eigenvalues have zero real parts. There exist matrices $T_a \in \mathbb{R}^{q_1 \times q_2}$, $F_a \in \mathbb{R}^{n \times q_2}$ solutions to the regulator equations:

$$\begin{align*}
-A_{11}T_a + T_aA_{22} + A_{12} &= -E_1F_a, \\
-Ce_1T_a + Ce_2 &= 0.
\end{align*}$$

Stability and regulation

If $\xi$ exp. converges to zero, then $\varepsilon(t) \to 0$. Furthermore, the trajectories are bounded.

Proof: If $C_0\xi$ converges to zero, then so does $\|(\alpha, \beta)\|_{L^2}$.

- We have

  $$\begin{align*}
  \dot{Y}_1 &= (A_{11} + E_1F_1)Y_1(t) + (A_{12} + E_1(F_a + F_1T_a))Y_2(t) + E_1\alpha(t, 1) \\
  &= (A_{11} + E_1F_1)Y_1(t) + (A_{11}T_a - E_1F_a - T_aA_{22})Y_2(t) + E_1(F_a + F_1T_a)Y_2(t) + E_1\alpha(t, 1),
  \end{align*}$$

  $\Rightarrow (Y_1 + T_aY_2)(t) = \bar{A}_{11}(Y_1 + T_aY_2) + E_1\alpha(t, 1)$.

- $Y_1 + T_aY_2$ exp. stable $\Rightarrow C_e(Y_1 + T_aY_2)(t) = C_{e1}Y_1(t) + C_{e2}Y_2(t) = \varepsilon(t)$ goes to zero.
A cascade structure

Assumption 6

The matrix $A_{22}$ is marginally stable, i.e., all its eigenvalues have zero real parts. There exist matrices $T_a \in \mathbb{R}^{q_1 \times q_2}, F_a \in \mathbb{R}^{n \times q_2}$ solutions to the regulator equations:

\[
\begin{aligned}
- A_{11} T_a + T_a A_{22} + A_{12} &= -E_1 F_a, \\
- C_{e1} T_a + C_{e2} &= 0.
\end{aligned}
\]

Stability and regulation

If $\xi$ exp. converges to zero, then $\varepsilon(t) \to 0$. Furthermore, the trajectories are bounded.

Proof: If $C_0 \xi$ converges to zero, then so does $\|(\alpha, \beta)\|_{L^2}$.

- We have

\[
\begin{aligned}
\dot{Y}_1 &= (A_{11} + E_1 F_1) Y_1(t) + (A_{12} + E_1 (F_a + F_1 T_a)) Y_2(t) + E_1 \alpha(t,1) \\
&= (A_{11} + E_1 F_1) Y_1(t) + (A_{11} T_a - E_1 F_a - T_a A_{22}) Y_2(t) + E_1 (F_a + F_1 T_a) Y_2(t) + E_1 \alpha(t,1),
\end{aligned}
\]

\[
\Rightarrow (Y_1 + T_a Y_2)(t) = \bar{A}_{11} (Y_1 + T_a Y_2) + E_1 \alpha(t,1) \xrightarrow{\to 0}
\]

- $Y_1 + T_a Y_2$ exp. stable $\Rightarrow C_e (Y_1 + T_a Y_2)(t) = C_{e1} Y_1(t) + C_{e2} Y_2(t) = \varepsilon(t)$ goes to zero.

- Invertibility + boundedness of the backstepping transf. implies boundedness of the state.
Time-delay representation

\[
\alpha_t(t, x) + \lambda \alpha_x(t, x) = 0
\]
\[
\beta_t(t, x) - \mu \beta_x(t, x) = 0
\]

\[
\alpha(t, 0) = q \beta(t, 0) + C_0 \xi(t)
\]
\[
\beta(t, 1) = \rho \alpha(t, 1)
\]
Time-delay representation

\[ \alpha_t(t, x) + \lambda \alpha_x(t, x) = 0 \rightarrow \text{Transport equation} \]
\[ \beta_t(t, x) - \mu \beta_x(t, x) = 0 \rightarrow \text{Transport equation} \]

\[ \alpha(t, 0) = q\beta(t, 0) + C_0 \xi(t) \]
\[ \beta(t, 1) = \rho \alpha(t, 1) \]
Time-delay representation

\[ \alpha_t(t, x) + \lambda \alpha_x(t, x) = 0 \rightarrow \text{Transport equation} \]
\[ \beta_t(t, x) - \mu \beta_x(t, x) = 0 \rightarrow \text{Transport equation} \]

\[ \alpha(t, 0) = q \beta(t, 0) + C_0 \xi(t) \]
\[ \beta(t, 1) = \rho \alpha(t, 1) \]

Method of characteristics:

\[ \alpha(t, x) = \alpha(t - \frac{x}{\lambda}, 0), \quad \beta(t, x) = \rho \alpha(t - \frac{(1 - x)}{\mu} - \frac{1}{\lambda}, 0) \]

**Difference Equation satisfied by \( \alpha(t, 0) \)**

\[ \alpha(t, 0) = \rho q \alpha(t - \tau, 0) + C_0 \xi(t), \quad t > \frac{1}{\lambda} + \frac{1}{\mu} = \tau \]

Using the Laplace transform: \( (1 - \rho q e^{-\tau s}) \alpha(s, 0) = C_0 \xi(s) \)

**We can kill the \( \alpha \) and \( \beta \) terms to obtain \( \xi \)-terms!**
Time-delay representation

\[ \dot{\eta}(t) = \bar{A}_1 \eta(t) + (E_1 \ 0)^\top \alpha(t, 1) \]
\[ \dot{\xi}(t) = \bar{A}_0 \xi(t) + \bar{E}_1 \alpha(t, 1) + \bar{E}_0 \beta(t, 0) + M\eta(t) + \int_0^1 M\alpha(y)\alpha(t, y) + M\beta(y)\beta(t, y)dy + B\bar{X}\bar{U}(t). \]

Laplace transform on \( \eta_1 \)

\[ \eta_1(s) = (sld - \bar{A}_{11})^{-1}(\bar{A}_{12}\eta_2(s) + E_1 e^{-\frac{s}{\lambda}}\alpha(s, 0)) \]

We can get rid of the \( \eta_1 \)-terms!
Time-delay representation

\[
\dot{\eta}(t) = \bar{A}_1 \eta(t) + (E_1 \ 0)^\top \alpha(t, 1)
\]

\[
\dot{\xi}(t) = \bar{A}_0 \xi(t) + \bar{E}_1 \alpha(t, 1) + \bar{E}_0 \beta(t, 0) + M \eta(t) + \int_0^1 M_\alpha(y) \alpha(t, y) + M_\beta(y) \beta(t, y) dy + B_X \bar{U}(t).
\]

Laplace transform on \( \eta_1 \)

\[
\eta_1(s) = (s \text{Id} - \bar{A}_{11})^{-1}(\bar{A}_{12} \eta_2(s) + E_1 e^{-\frac{s}{\lambda}} \alpha(s, 0))
\]

We can get rid of the \( \eta_1 \)-terms!

Laplace transform on \( \xi \)

\[
(s \text{Id} - \bar{A}_0) \xi(s) = G(s) C_0 \xi(s) + H(s) \eta_2(s) + B_X \bar{U}(s),
\]

\[
P_0 = C_0(s \text{Id} - \bar{A}_0)^{-1} B_X \text{ admits a stable right inverse } P_0^+.
\]

\[
C_0 \xi(s) = C_0(s \text{Id} - \bar{A}_0)^{-1} G(s) C_0 \xi(s) + C_0(s \text{Id} - \bar{A}_0)^{-1} H(s) \eta_2(s) + P_0(s) \bar{U}(s),
\]
Time-delay representation

\[
\begin{align*}
\dot{\eta}(t) &= \bar{A}_1 \eta(t) + (E_1 \ 0)^\top \alpha(t, 1) \\
\dot{\xi}(t) &= \bar{A}_0 \xi(t) + \bar{E}_1 \alpha(t, 1) + \bar{E}_0 \beta(t, 0) + M \eta(t) + \int_0^1 M_\alpha(y) \alpha(t, y) + M_\beta(y) \beta(t, y) \, dy + B_X \bar{U}(t).
\end{align*}
\]

Laplace transform on \(\eta_1\)

\[
\eta_1(s) = (s I - \bar{A}_{11})^{-1} \left( \bar{A}_{12} \eta_2(s) + E_1 e^{-\frac{s}{\lambda}} \alpha(s, 0) \right)
\]

We can get rid of the \(\eta_1\)-terms!

Laplace transform on \(\xi\)

\[
(s I - \bar{A}_0) \xi(s) = G(s) C_0 \xi(s) + H(s) \eta_2(s) + B_X \bar{U}(s),
\]

\(P_0 = C_0 (s I - \bar{A}_0)^{-1} B_X\) admits a stable right inverse \(P_0^+\).

\[
C_0 \xi(s) = C_0 (s I - \bar{A}_0)^{-1} G(s) C_0 \xi(s) + C_0 (s I - \bar{A}_0)^{-1} H(s) \eta_2(s) + P_0(s) \bar{U}(s),
\]

Stabilizing control law

\[
\bar{U}(s) = -P_0^+(s) C_0 (s I - \bar{A}_0)^{-1} G(s) C_0 \xi(s) - P_0^+(s) C_0 (s I - \bar{A}_0)^{-1} H(s) \eta_2(s)
\]

- \(\frac{\text{stabilization}}{\text{disturbance rejection or tracking}}\)
A non strictly proper control law

\[
\bar{U}(s) = -P_0^+(s)C_0(sI - \bar{A}_0)^{-1}G(s)C_0\xi(s) - P_0^+(s)C_0(sI - \bar{A}_0)^{-1}H(s)\eta_2(s)
\]

= \underbrace{F_\xi(s)\xi(s) + F_\eta(s)\eta_2(s)}_{\text{stabilization}}

\text{disturbance rejection or tracking}

- The control law ay not be \textbf{strictly proper} due to \(P_0^+(s) \rightarrow \) Robustness issues.

- We can make \(F_\eta(s)\) strictly proper using our prior knowledge of the dynamics.

- We can make \(F_\xi(s)\) strictly proper using a low-pass filter.
Filtering of the control input

\[ F_\xi(s) = -P_0^+(s)C_0(s\text{id} - \bar{A}_0)^{-1}G(s)C_0, \quad F_\eta(s) = -P_0^+(s)C_0(s\text{id} - \bar{A}_0)^{-1}H(s) \]

**Filtered control law**

Let \( w(s) \) be any low-pass filter, with a sufficiently high relative degree, and \( 0 < \delta < 1 \) such that

\[
\forall x \in \mathbb{R}, \quad |1 - w(jx)| \leq \frac{1 - \delta}{\|G\|_\infty \bar{\sigma}(C_0(jx\text{id} - \bar{A}_0)^{-1})},
\]

then \( \bar{U}(s) = w(s)F_\xi(s)\xi(s) + F_\eta(s)\eta_2(s) \) stabilizes \( C_0\xi(s) \)

**Proof:** Let \( \Phi(s) = (1 - w(s))C_0(s\text{id} - \bar{A}_0)^{-1}G(s) \).
Filtering of the control input

\[ F_\xi(s) = -P_0^+(s)C_0(sld - A_0)^{-1}G(s)C_0, \quad F_\eta(s) = -P_0^+(s)C_0(sld - A_0)^{-1}H(s) \]

Filtered control law

Let \( w(s) \) be any low-pass filter, with a sufficiently high relative degree, and \( 0 < \delta < 1 \) such that

\[ \forall x \in \mathbb{R}, \ |1 - w(jx)| \leq \frac{1 - \delta}{\|G\|_\infty \bar{\sigma}(C_0(jxld - A_0)^{-1})}, \]

then \( \bar{U}(s) = w(s)F_\xi(s)\xi(s) + \bar{F}_\eta(s)\eta_2(s) \) stabilizes \( C_0\xi(s) \)

**Proof:** Let \( \Phi(s) = (1 - w(s))C_0(sld - A_0)^{-1}G(s). \)

- \( \Phi \) is stable and strictly proper
Filtering of the control input

\[ F_\xi(s) = -P_0^+(s)C_0(sI_d - \bar{A}_0)^{-1}G(s)C_0, \quad F_\eta(s) = -P_0^+(s)C_0(sI_d - \bar{A}_0)^{-1}H(s) \]

Filtered control law

Let \( w(s) \) be any low-pass filter, with a sufficiently high relative degree, and \( 0 < \delta < 1 \) such that

\[ \forall x \in \mathbb{R}, |1 - w(jx)| \leq \frac{1 - \delta}{\|G\|_\infty \bar{\sigma}(C_0(jxI_d - \bar{A}_0)^{-1})}, \]

then \( \bar{U}(s) = w(s)F_\xi(s)\xi(s) + \bar{F}_\eta(s)\eta_2(s) \) stabilizes \( C_0\xi(s) \)

Proof: Let \( \Phi(s) = (1 - w(s))C_0(sI_d - \bar{A}_0)^{-1}G(s) \).

- \( \Phi \) is stable and strictly proper
- \( G(s) \) is unif. bounded, we have \( \bar{\sigma}(G(jx)) \leq \|G\|_\infty \) for all \( x \)
Filtering of the control input

\[ F_\xi(s) = -P_0^+(s)C_0(sI - \tilde{A}_0)^{-1}G(s)C_0, \quad F_\eta(s) = -P_0^+(s)C_0(sI - \tilde{A}_0)^{-1}H(s) \]

Filtered control law

Let \( w(s) \) be any low-pass filter, with a sufficiently high relative degree, and \( 0 < \delta < 1 \) such that

\[ \forall x \in \mathbb{R}, \quad |1 - w(jx)| \leq \frac{1 - \delta}{\|G\|_{\infty} \tilde{\sigma}(C_0(jxI - \tilde{A}_0)^{-1})}, \]

then \( \tilde{U}(s) = w(s)F_\xi(s)\xi(s) + F_\eta(s)\eta_2(s) \) stabilizes \( C_0\xi(s) \)

**Proof:** Let \( \Phi(s) = (1 - w(s))C_0(sI - \tilde{A}_0)^{-1}G(s) \).

- \( \Phi \) is stable and strictly proper
- \( G(s) \) is unif. bounded, we have \( \tilde{\sigma}(G(jx)) \leq \|G\|_{\infty} \) for all \( x \)
- We have \( \tilde{\sigma}(\phi(jx)) \leq 1 - \delta \Rightarrow \|\Phi\|_{\infty} < 1 \)
Filtering of the control input

\[
F_\xi(s) = -P_0^+(s)C_0(sI - \bar{A}_0)^{-1}G(s)C_0, \quad F_\eta(s) = -P_0^+(s)C_0(sI - \bar{A}_0)^{-1}H(s)
\]

Filtered control law

Let \( w(s) \) be any low-pass filter, with a sufficiently high relative degree, and \( 0 < \delta < 1 \) such that

\[
\forall x \in \mathbb{R}, \quad |1 - w(jx)| \leq \frac{1 - \delta}{\|G\|_\infty \bar{\sigma}(C_0(sI - \bar{A}_0)^{-1})},
\]

then \( \bar{U}(s) = w(s)F_\xi(s)\xi(s) + F_\eta(s)\eta_2(s) \) stabilizes \( C_0\xi(s) \)

Proof: Let \( \Phi(s) = (1 - w(s))C_0(sI - \bar{A}_0)^{-1}G(s) \).

- \( \Phi \) is stable and strictly proper
- \( G(s) \) is unif. bounded, we have \( \bar{\sigma}(G(jx)) \leq \|G\|_\infty \) for all \( x \)
- We have \( \bar{\sigma}(\phi(jx)) \leq 1 - \delta \Rightarrow \|\Phi\|_\infty < 1 \)
- Characteristic equation \( (1 - \Phi(s))C_0\xi(s) = 0 \) \( \Rightarrow \) exponential stability
Filtering of the control input

\[ F_\xi(s) = -P_0^+(s)C_0(s\text{Id} - \bar{A}_0)^{-1}G(s)C_0, \quad F_\eta(s) = -P_0^+(s)C_0(s\text{Id} - \bar{A}_0)^{-1}H(s) \]

Filtered control law

Let \( w(s) \) be any low-pass filter, with a sufficiently high relative degree, and \( 0 < \delta < 1 \) such that

\[
\forall x \in \mathbb{R}, \quad |1 - w(jx)| \leq \frac{1 - \delta}{\|G\|_\infty \bar{\sigma}(C_0(jx\text{Id} - \bar{A}_0)^{-1})}
\]

then \( \tilde{U}(s) = w(s)F_\xi(s)\xi(s) + F_\eta(s)\eta_2(s) \) stabilizes \( C_0\xi(s) \)

Proof: Let \( \Phi(s) = (1 - w(s))C_0(s\text{Id} - \bar{A}_0)^{-1}G(s) \).

- \( \Phi \) is stable and strictly proper
- \( G(s) \) is unif. bounded, we have \( \bar{\sigma}(G(jx)) \leq \|G\|_\infty \) for all \( x \)
- We have \( \bar{\sigma}(\phi(jx)) \leq 1 - \delta \Rightarrow \|\Phi\|_\infty < 1 \)
- Characteristic equation \((1 - \Phi(s))C_0\xi(s) = 0 \) → exponential stability

Strictly proper stabilizing control law!
Summary state-feedback

- Backstepping transformation to simplify the dynamics and the design of the control law.

- The regulation problem rewrites as a stabilization problem.

- Time-delay representation and frequency analysis.

- Low-pass filtering of the control law to make it strictly proper.
Observer design

\[
\begin{align*}
\dot{X}(t) &= A_0 X(t) + E_0 v(t, 0) + B_X U(t), \\
\partial_t u(t, x) + \Lambda^+ \partial_x u(t, x) &= \Sigma^{++}(x) u(t, x) + \Sigma^{+-}(x) v(t, x), \\
\partial_t v(t, x) - \Lambda^- \partial_x v(t, x) &= \Sigma^{-+}(x) u(t, x) + \Sigma^{--}(x) v(t, x), \\
u(t, 0) &= C_0 X(t) + Q v(t, 0), \quad v(t, 1) = R u(t, 1) + C_1 Y(t), \\
\dot{Y}(t) &= A_{11} Y(t) + E_1 u(t, 1), \\
y &= C_{mes} Y(t), \quad \dim(y) \geq \dim(u)
\end{align*}
\]

Problem statement

Design a **state observer** for the system based on the available measurement \( y(t) \).
Methodology

- Backstepping transformation to simplify the dynamics and the design of the observer.
Methodology

- Backstepping transformation to simplify the dynamics and the design of the observer.

- Luenberger-like observer with operators $O_i$ that need to be tuned.
Methodology

- Backstepping transformation to simplify the dynamics and the design of the observer.
- Luenberger-like observer with operators $O_i$ that need to be tuned.
- Design of the operators $O_i$ to guarantee the exponential stability of the error system.
Methodology

- Backstepping transformation to simplify the dynamics and the design of the observer.

- Luenberger-like observer with operators $O_i$ that need to be tuned.

- Design of the operators $O_i$ to guarantee the exponential stability of the error system.

- Convergence of the observer state to the real state.
Backstepping: Target system

Original system:

\[
\dot{X} = A_0 X + E_0 v(t, 0) + B_X U(t)
\]

Target system

\[
\dot{\xi} = \tilde{A}_0 \xi + G_3 \alpha(t, 1) + G_4 Y + B_X U(t)
\]
Backstepping: Target system

Target system

\[ \dot{\xi} = \tilde{A}_0 \xi + G_3 \alpha(t, 1) + G_4 Y + B_X U(t) \]

\[ \dot{Y}(t) = A_1 Y(t) + E_1 \alpha(t, 1) \]

\[ \dot{\xi}(t) = \tilde{A}_0 \xi(t) + G_3 \alpha(t, 1) + G_4 Y(t) + B_X U(t), \]

\[ \alpha(t, 0) = Q \beta(t, 0) + C_0 \xi(t) + (Q \gamma_\beta(0) - \gamma_\alpha(0)) Y(t) + \int_0^1 F^\alpha(y) \alpha(t, y) + F^\beta(y) \beta(t, y) dy, \]

\[ \alpha_t(t, x) + \Lambda^+ \alpha_x(t, x) = G_1(x) \alpha(t, 1), \]

\[ \beta_t(t, x) - \Lambda^- \beta_x(t, x) = G_2(x) \alpha(t, 1), \]

\[ \beta(t, 1) = R \alpha(t, 1), \quad \dot{Y}(t) = A_1 Y(t) + E_1 \alpha(t, 1). \]

\( F^\alpha \) strictly lower triangular
Backstepping: Target system

Target system

\[
\dot{\xi} = \tilde{A}_0 \xi + G_3 \alpha(t, 1) + G_4 Y + B_x U(t)
\]

Advantages of the target system:
- Simplified in-domain couplings.
- Almost a "cascade structure" (except for the $\alpha(t, 1)$-terms);
- Simplified observer design
Backstepping: Volterra transformation

\[ X(t) = \xi(t) - \int_0^1 L_1(y)\alpha(y) + L_2(y)\beta(y) \, dy, \]

\[ u(t, x) = \alpha(t, x) - \int_x^1 L^{\alpha\alpha}(x, y)\alpha(y) \, dy - \int_x^1 L^{\alpha\beta}(x, y)\beta(y) \, dy + \gamma_\alpha(x)Y(t), \]

\[ v(t, x) = \beta(t, x) - \int_x^1 L^{\beta\alpha}(x, y)\alpha(y) \, dy - \int_x^1 L^{\beta\beta}(x, y)\beta(y) \, dy + \gamma_\beta(x)Y(t), \]

\[ Y(t) = Y(t), \]

- Triangular transformation: invertible.

- Kernels are bounded functions.
Observer equations

System \((\xi, \alpha, \beta, Y)\)

\[
\dot{\xi}(t) = \tilde{A}_0 \xi(t) + G_3 \alpha(t, 1) + G_4 Y(t) + B_X U(t),
\]

\[
\alpha(t, 0) = Q\beta(t, 0) + C_0 \xi(t) + (Q\gamma_\beta(0) - \gamma\alpha(0)) Y(t) + \int_0^1 F^\alpha(y)\alpha(t, y) + F^\beta(y)\beta(t, y)dy,
\]

\[
\alpha_t(t, x) + \Lambda^+\alpha_x(t, x) = G_1(x)\alpha(t, 1),
\]

\[
\beta_t(t, x) - \Lambda^-\beta_x(t, x) = G_2(x)\alpha(t, 1),
\]

\[
\beta(t, 1) = R\alpha(t, 1), \quad \dot{Y}(t) = A_1 Y(t) + E_1 \alpha(t, 1).
\]

System \((\hat{\xi}, \hat{\alpha}, \hat{\beta}, \hat{Y})\):

\(O_1: \text{stable operators.}\)

\[
\dot{\hat{\xi}}(t) = \tilde{A}_0 \hat{\xi}(t) + G_3 \hat{\alpha}(t, 1) + G_4 \hat{Y}(t) - O_0(\tilde{\gamma}),
\]

\[
\hat{\alpha}(t, 0) = Q\hat{\beta}(t, 0) + C_0 \hat{\xi}(t) + (Q\gamma_\beta(0) - \gamma\alpha(0)) \hat{Y}(t)
\]

\[
+ \int_0^1 F^\alpha(y)\hat{\alpha}(t, y) + F^\beta(y)\hat{\beta}(t, y)dy - O_1(\tilde{\gamma}),
\]

\[
\hat{\alpha}_t(t, x) + \Lambda^+\hat{\alpha}_x(t, x) = G_1(x)\hat{\alpha}(t, 1) - O_\alpha(x, \tilde{\gamma}),
\]

\[
\hat{\beta}_t(t, x) - \Lambda^-\hat{\beta}_x(t, x) = G_2(x)\hat{\alpha}(t, 1) - O_\beta(x, \tilde{\gamma}),
\]

\[
\hat{\beta}(t, 1) = R\hat{\alpha}(t, 1), \quad \dot{\hat{Y}}(t) = A_1 \hat{Y}(t) + E_1 \hat{\alpha}(t, 1) - L_1 C\tilde{\gamma},
\]
Error system

\[\ddot{\xi}(t) = \tilde{A}_0 \dot{\xi}(t) + G_3 \tilde{\alpha}(t, 1) + G_4 \dot{Y}(t) + B_X U(t) O_0(\tilde{y}),\]
\[\tilde{\alpha}(t, 0) = C_0 \dot{\xi}(t) + Q \tilde{\beta}(t, 0) + (Q \gamma_\beta(0) - \gamma_\alpha(0)) \dot{Y}(t)\]
\[+ \int_0^1 F^\alpha(y) \tilde{\alpha}(t, y) + F^\beta(y) \tilde{\beta}(t, y) dy + O_1(\tilde{y}),\]
\[\tilde{\alpha}_t(t, x) + \Lambda^+ \tilde{\alpha}_x(t, x) = G_1(x) \tilde{\alpha}(t, 1) + O_\alpha(x, \tilde{y})\]
\[\tilde{\beta}_t(t, x) - \Lambda^- \tilde{\beta}_x(t, x) = G_2(x) \tilde{\alpha}(t, 1) + O_\beta(x, \tilde{y})\]
\[\tilde{\beta}(t, 1) = R \tilde{\alpha}(t, 1), \quad \dot{Y}(t) = \tilde{A}_1 \dot{Y}(t) + E_1 \tilde{\alpha}(t, 1).\]

**Objective:** Tune the gains \(O_i\) such that the error system exponentially converges to zero.

**Lemma:** Cascade structure of the error system

If \(\ddot{\xi}(t), \tilde{\alpha}(t, 1)\) and \(\dot{Y}(t)\) exponentially converge to zero, then the state \((\ddot{\xi}, \tilde{\alpha}, \tilde{\beta}, \dot{Y})\) exponentially converges to zero. This implies the convergence of the observer state to the real state.
Design of the operators $O_i$

- Laplace transform of $\dot{Y}(t) = \tilde{A}_1 \dot{Y}(t) + E_1 \tilde{\alpha}(t, 1)$:

  $$(\text{sld} - \tilde{A}_1) \tilde{Y}(s) = E_1 \tilde{\alpha}(s, 1) \Rightarrow \tilde{y}(s) = C_{\text{mes}}(\text{sld} - \tilde{A}_1)^{-1} E_1 \tilde{\alpha}(s, 1),$$

where $\tilde{A}_1$ is Hurwitz (Assumption 4) and $C_{\text{mes}}(\text{sld} - \tilde{A}_1)^{-1} E_1$ has no zeros in the RHP (Assumption 2)
Design of the operators $O_i$

- Laplace transform of $\dot{\tilde{Y}}(t) = \tilde{A}_1 \tilde{Y}(t) + E_1 \tilde{\alpha}(t, 1)$:

  $$(\text{sl}d - \tilde{A}_1) \tilde{Y}(s) = E_1 \tilde{\alpha}(s, 1) \implies \tilde{y}(s) = C_{\text{mes}} (\text{sl}d - \tilde{A}_1)^{-1} E_1 \tilde{\alpha}(s, 1),$$

  where $\tilde{A}_1$ is Hurwitz (Assumption 4) and $C_{\text{mes}} (\text{sl}d - \tilde{A}_1)^{-1} E_1$ has no zeros in the RHP (Assumption 2).

- $P_1(s) = C_{\text{mes}} (\text{sl}d - \tilde{A}_1)^{-1} E_1$ has a stable left-inverse (Assumption 4):

  $\tilde{\alpha}(s, 1) = P_1^-(s) \tilde{y}(s), \quad \tilde{Y}(s) = (\text{sl}d - \tilde{A}_1)^{-1} E_1 P_1^-(s) \tilde{y}(s)$

  Terms that are functions $\tilde{Y}$ and $\tilde{\alpha}(s, 1)$ can be (exponentially) compensated using stable filters and values of $\tilde{y}(s)$. 

Terms that are functions $\tilde{Y}$ and $\tilde{\alpha}(s, 1)$ can be (exponentially) compensated using stable filters and values of $\tilde{y}(s)$. 

-pseudo code-
Design of the operators $O_i$

- Laplace transform of $\dot{\tilde{Y}}(t) = \tilde{\alpha}_1 \tilde{Y}(t) + E_1 \tilde{\alpha}(t, 1)$:

$$ (sld - \tilde{\alpha}_1) \tilde{Y}(s) = E_1 \tilde{\alpha}(s, 1) \rightarrow \tilde{y}(s) = C_{mes}(sld - \tilde{\alpha}_1)^{-1} E_1 \tilde{\alpha}(s, 1), $$

where $\tilde{\alpha}_1$ is Hurwitz (Assumption 4) and $C_{mes}(sld - \tilde{\alpha}_1)^{-1} E_1$ has no zeros in the RHP (Assumption 2)

- $P_1(s) = C_{mes}(sld - \tilde{\alpha}_1)^{-1} E_1$ has a stable left-inverse (Assumption 4):

$$ \tilde{\alpha}(s, 1) = P_1^{-1}(s) \tilde{y}(s), \quad \tilde{Y}(s) = (sld - \tilde{\alpha}_1)^{-1} E_1 P_1^{-1}(s) \tilde{y}(s) $$

Terms that are functions $\tilde{Y}$ and $\tilde{\alpha}(s, 1)$ can be (exponentially) compensated using stable filters and values of $\tilde{y}(s)$.

- We have $\dot{\tilde{\xi}}(t) = \tilde{\alpha}_0 \tilde{\xi}(t) + G_3 \tilde{\alpha}(t, 1) + G_4 \tilde{Y}(t) + O_0(\tilde{y})$

$$ O_0(\tilde{y}(s)) = -(G_3 P_1^{-1}(s) + G_4 (sld - \tilde{\alpha}_1)^{-1} E_1 P_1^{-1}(s)) \tilde{y}(s) \Rightarrow (sld - \tilde{\alpha}_0) \tilde{\xi}(s) = 0 $$

Exponential convergence of $\tilde{\xi}$ to 0.
Design of the operators $O_i$

\[ \alpha(s, 1) = P^-_1(s)\tilde{Y}(s), \quad \tilde{y}(s) = (sld - \tilde{A}_1)^{-1}E_1P^-_1(s)\tilde{y}(s) \]

- We have $\tilde{\alpha}_t(t, x) + \Lambda^+ \tilde{\alpha}_x(t, x) = G_1(x)\tilde{\alpha}(t, 1) + O_\alpha(x, \tilde{y})$. Thus

\[ O_\alpha(x, \tilde{y}) = -G_1(x)P^-_1(s)\tilde{y}(s) \Rightarrow \tilde{\alpha}_t(t, x) + \Lambda^+ \tilde{\alpha}_x(t, x) = 0 \Rightarrow \tilde{\alpha}_i(t, x) = \tilde{\alpha}_i(t - \frac{x}{\lambda_i}, 0). \]
Design of the operators $O_i$

\[ \alpha(s, 1) = P_1^-(s)\tilde{Y}(s), \quad \tilde{y}(s) = (s\text{Id} - \tilde{A}_1)^{-1}E_1P_1^-(s)\tilde{y}(s) \]

- We have $\tilde{\alpha}_t(t, x) + \Lambda^+\tilde{\alpha}_x(t, x) = G_1(x)\tilde{\alpha}(t, 1) + O_\alpha(x, \tilde{y})$. Thus

\[ O_\alpha(x, \tilde{y}) = -G_1(x)P_1^-(s)\tilde{y}(s) \Rightarrow \tilde{\alpha}_t(t, x) + \Lambda^+\tilde{\alpha}_x(t, x) = 0 \Rightarrow \tilde{\alpha}_i(t, x) = \tilde{\alpha}_i(t - \frac{x}{\lambda_i}, 0). \]

- We have $\tilde{\beta}_t(t, x) - \Lambda^-\tilde{\beta}_x(t, x) = G_2(x)\tilde{\alpha}(t, 1) + O_\beta(x, \tilde{y})$. Thus

\[ O_\beta(x, \tilde{y}) = -G_2(x)P_1^-(s)\tilde{y}(s) \Rightarrow \tilde{\beta}_t(t, x) - \Lambda^-\tilde{\beta}_x(t, x) = 0 \Rightarrow \beta_j(t, x) = \sum_{k=1}^n R_{jk}\tilde{\alpha}_k(t - \frac{1 - x}{\mu_j}, 1). \]
The function $\tilde{\alpha}(t, 0)$ verifies

$$\tilde{\alpha}_i(s, 0) = ((Q\gamma_\beta(0) - \gamma_\alpha(0))\tilde{Y})_i + (O_1(\tilde{y}))_i + \sum_{k=1}^{m} \sum_{\ell=1}^{n} Q_{ik} R_{k\ell} e^{-\frac{s}{\mu_k} - \frac{s}{\lambda_\ell}} \tilde{\alpha}_\ell(s, 0)$$

$$+ \int_0^1 \sum_{k=1}^{m} \sum_{\ell=1}^{n} F_{ik}^\beta(v) R_{k\ell} e^{-\frac{s(v(1-v)}{\mu_k}} \tilde{\alpha}_\ell(s, 1) dv$$

$$+ \int_0^1 \sum_{j=1}^{i} \sum_{k=1}^{m} \sum_{\ell=1}^{n} Q_{jk} R_{k\ell} e^{-\frac{sv}{\lambda_\ell} + \frac{s}{\mu_k}} \tilde{\alpha}_\ell(s, 1) dv,$$

since $F^\alpha$ is strictly lower-triangular.
The function $\tilde{\alpha}(t, 0)$ verifies

$$
\tilde{\alpha}_i(s, 0) = ((Q\gamma_\beta(0) - \gamma_\alpha(0))\tilde{Y})_i + (O_1(\tilde{y}))_i + \sum_{k=1}^{m} \sum_{\ell=1}^{n} Q_{ik} R_{k\ell} e^{-\frac{s}{\mu_k}} - \frac{s}{\lambda_\ell} \tilde{\alpha}_\ell(s, 0)
$$

$$
+ \int_0^1 \sum_{k=1}^{m} \sum_{\ell=1}^{n} F^\beta_{ik}(\nu) R_{k\ell} e^{-\frac{s(1-\nu)}{\mu_k}} \tilde{\alpha}_\ell(s, 1) d\nu
$$

$$
+ \int_0^1 \sum_{i=1}^{i} \sum_{j=1}^{m} F^\alpha_{ij}(\nu) \sum_{k=1}^{m} \sum_{\ell=1}^{n} Q_{jk} R_{k\ell} e^{-\frac{sv}{\lambda_j}} - \frac{s}{\mu_k} \tilde{\alpha}_\ell(s, 1) d\nu,
$$

since $F^\alpha$ is strictly lower-triangular.

Possible to recursively define $O_1(\tilde{y})$ such that

$$
\tilde{\alpha}_i(t, 0) = \sum_{k=1}^{m} \sum_{\ell=1}^{n} Q_{ik} R_{k\ell} \tilde{\alpha}_\ell(t - \frac{1}{\mu_k} - \frac{1}{\lambda_\ell}, 0)
$$
An Integral Difference Equation

The function $\tilde{\alpha}(t, 0)$ verifies

$$\tilde{\alpha}_i(s, 0) = ((Q\gamma_\beta(0) - \gamma_\alpha(0))\tilde{Y})_i + (O_1(\tilde{y}))_i + \sum_{k=1}^{m} \sum_{\ell=1}^{n} Q_{ik} R_{k\ell} e^{-\frac{s}{\mu_k}} e^{-\frac{s}{\lambda_\ell}} \tilde{\alpha}_\ell(s, 0)$$

$$+ \int_0^1 \sum_{k=1}^{m} \sum_{\ell=1}^{n} F_{ik}(\nu) R_{k\ell} e^{-\frac{s(1-\nu)}{\mu_k}} \tilde{\alpha}_\ell(s, 1) d\nu$$

$$+ \int_0^1 \sum_{j=1}^{i} F_{ij}^\alpha(\nu) \sum_{k=1}^{m} \sum_{\ell=1}^{n} Q_{jk} R_{k\ell} e^{-\frac{s\nu}{\lambda_j}} e^{-\frac{s}{\mu_k}} \tilde{\alpha}_\ell(s, 1) d\nu,$$

since $F^\alpha$ is strictly lower-triangular.

Possible to recursively define $O_1(\tilde{y})$ such that

$$\tilde{\alpha}_i(t, 0) = \sum_{k=1}^{m} \sum_{\ell=1}^{n} Q_{ik} R_{k\ell} \tilde{\alpha}_\ell(t - \frac{1}{\mu_k} - \frac{1}{\lambda_\ell}, 0)$$

Exponential stabilization of $\tilde{\alpha}(t, 0)$ (and consequently of $\tilde{\alpha}(t, 1)$) due to Assumption 3.
Convergence of the observer

- The states $\tilde{\alpha}(t, 1)$ and $\tilde{\xi}$ exponentially converge to zero.

- We have $\dot{\tilde{Y}}(t) = \tilde{A}_1 \tilde{Y}(t) + E_1 \tilde{\alpha}(t, 1)$ with $\tilde{A}_1$ Hurwitz. Thus the state $\tilde{Y}$ exponentially converges to zero.

- Stabilization of the error system.

Convergence of the observer

With the proposed operators $O_0$, $O_\alpha$, $O_\beta$, $O_1$, the observer state $(\hat{X}, \hat{u}, \hat{v}, \hat{\bar{Y}}) = T(\hat{\xi}, \hat{\alpha}, \hat{\beta}, \hat{\bar{Y}})$ exponentially converges to $(X, u, v, Y)$, $T$ being the inverse backstepping transformation.
Convergence of the observer

• The states $\tilde{\alpha}(t, 1)$ and $\tilde{\xi}$ exponentially converge to zero.

• We have $\dot{\tilde{Y}}(t) = \tilde{A}_1 \tilde{Y}(t) + E_1 \tilde{\alpha}(t, 1)$ with $\tilde{A}_1$ Hurwitz. Thus the state $\tilde{Y}$ exponentially converges to zero.

• Stabilization of the error system.

Convergence of the observer

With the proposed operators $O_0$, $O_\alpha$, $O_\beta$, $O_1$, the observer state $(\hat{X}, \hat{u}, \hat{v}, \hat{Y}) = \mathcal{T}(\hat{\xi}, \hat{\alpha}, \hat{\beta}, \hat{Y})$ exponentially converges to $(X, u, v, Y)$, $\mathcal{T}$ being the inverse backstepping transformation.

• Possible to low-pass filter the measured output signal to use strictly proper observer operators
Convergence of the observer

- The states $\tilde{\alpha}(t, 1)$ and $\tilde{\xi}$ exponentially converge to zero.

- We have $\dot{\tilde{Y}}(t) = \tilde{\Lambda}_1 \tilde{Y}(t) + E_1 \tilde{\alpha}(t, 1)$ with $\tilde{\Lambda}_1$ Hurwitz. Thus the state $\tilde{Y}$ exponentially converges to zero.

- Stabilization of the error system.

Convergence of the observer

With the proposed operators $O_0, O_\alpha, O_\beta, O_1$, the observer state $(\hat{X}, \hat{u}, \hat{v}, \hat{Y}) = T(\hat{\xi}, \hat{\alpha}, \hat{\beta}, \hat{Y})$ exponentially converges to $(X, u, v, Y)$, $T$ being the inverse backstepping transformation.

- Possible to low-pass filter the measured output signal to use strictly proper observer operators

- The proposed observer could be combined with the previous state-feedback laws to obtain a strictly proper output-feedback controller.
Simulation results

Parameters:
\( \lambda = 2, \mu = 0.7, \sigma^+ = 1, \sigma^- = 0.5, \rho = 0.5, q = 1.2. \)

ODE dynamics in dimension \( n = 4, m = 3, c = 2 \)

\[
A_0 = \begin{bmatrix}
0 & 0.14 & 0 & 0.1 \\
0 & 0 & 0.14 & 0 \\
0.29 & -0.43 & 0.57 & 0.2 \\
0 & 0 & 0 & -1.1
\end{bmatrix},
B_0 = \begin{bmatrix}
0 & 0 \\
0 & -1 \\
1 & -1 \\
0 & 0
\end{bmatrix},
\]

\[
C_0 = \begin{bmatrix}
1 \\
0 \\
0 \\
-0.5
\end{bmatrix}^T, E_0 = \begin{bmatrix}
2 \\
-1 \\
0.1 \\
0
\end{bmatrix}, C_{11} = \begin{bmatrix}
0 \\
1 \\
0.5
\end{bmatrix}^T
\]

\[
A_{11} = \begin{bmatrix}
0.29 & 0.14 & 0 \\
0.14 & 0 & 0.1 \\
0 & 0 & -0.9
\end{bmatrix}, E_1 = \begin{bmatrix}
-1 \\
1 \\
0
\end{bmatrix}.
\]

We want to reject a sinusoidal disturbance

Unstable system in open-loop.
Simulation results

Figure: Evolution of the distal ODE state $Y_1(t)$ (blue) in the presence of a disturbance $Y_{\text{dist}}$
Figure: Evolution of the control inputs $U_1(t)$ (blue) and $U_2(t)$ (red)
Simulation results

**Figure:** Evolution of the PDE state \( v(t,x) \)
Simulation results

Figure: Evolution of the norm of the error state
Conclusions and perspectives

- **Strictly proper dynamic** state-feedback controller for dist. rejection and trajectory tracking
  - Backstepping transformation to simplify the structure of the system
  - Frequency analysis to design the control law
  - Filtering techniques to guarantee robustness
Conclusions and perspectives

- **Strictly proper dynamic** state-feedback controller for dist. rejection and trajectory tracking
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- **Luenberger-like observer** for the ODE-PDE-ODE system
  - Backstepping transformation and frequency analysis approach for the error system.
  - Output-feedback control law.
  - Computational effort?

Perspectives?

- Model reduction?
- Leverage the different assumptions?
- Structure of the interconnection?
Conclusions and perspectives

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- **Perspectives?**
  - Model reduction?
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